UNIVERSITY OF LEICESTER

PGCE Secondary Course Mathematics Subject Guide 2023-2024

MATHEMATICS Guide

This guide is designed to complement the course handbook; this document details aspects that refer specifically to mathematics.

Contents

1. Course Structure and Procedures

1.1 University Staff

Dr Fay Baldry

t: +44 (0)116 252 3671
e: fb128@le.ac.uk

Rachel Hunt
(Working Hours: Fridays 0830-1600hrs)
e: rh456@le.ac.uk

School of Education, University of Leicester,
21 University Road, Leicester, LE1 7RF, UK

1.2 Overview

1.2.1 Aims

The mathematics course aims to:

- Broaden and deepen personal knowledge and understanding of mathematics for teaching;
- Develop understanding and insight into the ways in which mathematics is learnt and potential difficulties;
- Provide a framework for interrogating curriculum requirements and the potential of technology;
- Raise awareness and provide experience of a broad range of teaching and learning approaches mathematics so all learners can be effectively supported, including highlighting the needs of students with learning difficulties;
- Develop skills in planning, communication, assessment, recording and evaluation;
- Develop an understanding of mathematics education research as it applies to the classroom: how it is carried out, how it can be applied and how it can be critiqued;
- Progressively develop confidence and competence in mathematics and its teaching, including the promotion of enthusiasm for and positive attitudes towards the educational potential of mathematics.
- Understand career development trajectories through the CCF (Core Content Framework) and ECF (Early Careers Framework)

1.2.2 Getting Started - Professional Expectations

The PGCE course is first year of your teaching career. The professional standards expected of PGCE students start as soon as the course commences.

Attendance requirements align with schools' expectations for teaching staff and you must follow the guidance set out in the course handbook. In summary, you are required to attend all parts of the university-based course, complete all the tasks set and attend all school-based practicum days. If through emergency or illness you are unable to attend, you need to contact secpgce@le.ac.uk and your subject tutor(s) for university-based days and secpgce@le.ac.uk ***and** your school* for school-based days. Any planned absence needs to be *requested* and you must follow the guidance set out in the course handbook.

1.2.3 Subject Sessions

The focus is on developing an understanding of the teaching and learning of mathematics, and how teachers can provide students with appropriate learning opportunities. This will involve direct engagement with mathematical activities, combined with an exploration of pedagogical research, to build links between theory and practice. *You are responsible for developing your subject knowledge…*

1.2.4 Teaching and Learning Methods

The mathematics course seeks to develop student-teachers' learning through a variety of means:

- Subject group sessions led by a university tutor or invited guest speakers. These will be supported with access to appropriate session resources and pre/post-session recommended reading. The sessions will include:
  - Collaborative student-teacher group and pair work
  - Student-teacher led sessions
    - Including peer-assessment
- Resource-based learning, involving the course handbook, videos of lessons and other resources
- Subject group visits to a school/college, if appropriate, for small-group teaching

1.3 Course Materials

The majority of materials used during university-based sessions are made available through 'Blackboard' (UoL VLE). Students are responsible for managing their use of these materials; these are designed to support the sessions, and are not usually intended for self-study.

1.3.1 Sources of information and resources

Reading List and Other Sources

The following book offers a starting point for professional reading.

> Watson, A., Jones, K. and Pratt, D., (2013). *Key Ideas in Teaching Mathematics: Research-based guidance for ages 9-19*. OUP Oxford.

You will also need to engage with academic texts; a full reading list is found on the Secondary PGCE Mathematics Blackboard site, which includes journals that focus on mathematics education. For example:

> Research in Mathematics Education; Journal of Mathematics Teacher Education; Educational Studies in Mathematics

Podcasts by Anne Watson http://www.pmtheta.com/annes-podcasts.html

Government Guidance

The DfE has published guidance on the teaching of mathematics

https://www.gov.uk/government/publications/teaching-mathematics-at-key-stage-3

NCETM

Government funded National Centre for Excellence in the Teaching of Mathematics https://www.ncetm.org.uk/

The Education Endowment Foundation (EEF)

This is a charity set up in 2011 with a stated aim of "breaking the link between family income and educational achievement". It has received significant funding from the government. The website provides accessible research summaries.

https://educationendowmentfoundation.org.uk/

Professional Organisations

There are materials and journals published by organisations focussed on the teaching of mathematics:

> Association of Teachers of Mathematics (ATM): *Mathematics Teaching* (see reading list for access to the journal and their website for further information)
>
> Mathematical Association (MA): *Mathematical Gazette* (mathematics for its own sake), *Mathematics in School* (relating to secondary schools), *Equal*s (low attainers).

Software

Dynamic geometry and graphing. Free access with many online tutorials are available:

> Geogebra https://www.geogebra.org
>
> Autograph https://completemaths.com/autograph

A website that includes online learning tools

> Desmos https://www.desmos.com/

Grid Algebra that links number to algebra

> https://gridalgebra.com/welcome

1.4 Course Components

1.4.1 Subject Knowledge: Audit

A GCSE and Core A-level specifications is provided as a starting point for your subject knowledge audit. However, you need to consider the whole of key stage 3 and the GCSE and A-level syllabi. In addition, the syllabi do vary and are liable to change, so you do need to look beyond these lists.

There are two main aspects to 'Subject Knowledge':

Your personal understanding (subject knowledge)

The teaching and learning of a topic (pedagogical content knowledge)

Track both of these aspects as you progress through the course. (Some find it more difficult to plan for topics they find intuitively 'easy', especially w.r.t. developing students' understanding).

Remember, the PGCE sessions focus on pedagogical understanding; *you need to take responsibility for developing your subject knowledge*. Keep a record of the work you do; this can relate to concepts met in University of Leicester (UoL) subject sessions and lessons taught on school practicums. For example, you should update your audit as part of your review (evaluation) of a topic that you have taught in school.

1.4.2 The University Assignments

There is more information about the University Assignments in the course handbook.

University Assignment 1 (UA1) has a common structure for all subjects on the Secondary PGCE. This focusses on learning and pedagogy from a theoretical perspective. Whilst this can be undertaken from a general stance about learning, you are advised to take a mathematics perspective for at least some of the assignment. This would include mathematics specific literature, focussing on wider issues in mathematics education and potentially topic specific literature (e.g. a wider issue - the role of questioning; a topic - fractions).

University Assignment 2 (UA2) also has a common structure for all subjects on the Secondary PGCE. This assignment focusses on your understanding of mathematical activity (a task, a sequence of tasks or activities) and the learning potential of those activities. The second strand is to view this activity through an EDI lens, considering the pupil experience.

You must engage with mathematics specific literature, including appropriate topic specific literature.

University Assignment 3 (UA3) is the additional 30 credit module that all students have the opportunity to complete if UA1 and UA2 are completed at level 7.

1.4.3 Directed Tasks

There will be a number of tasks set during the year that will contribute directly or indirectly to the University of Leicester assignments and/or meeting the Teachers' Standards. Any tasks set must be completed.

Assessing student progress

For example, you should track student progress in different ways:

In conjunction with considering the overall progress of your class(es), it is useful to consider the progress, over time, of 2-4 students. This detail can be linked to a particular medium-term plan, lesson plan(s) and resources that you have written/used. This can develop your understanding whilst also providing concrete examples for you to discuss with your co-tutor/mentor(s) and can act as points for reflection.

You should draw together a range of evidence relating to students' progress and articulate what this might mean in terms of their learning. This should include prior attainment data and progress data from the school, as well as your own monitoring of student progress. It should also include assessments information, interrogation of student work, both in class and at home, and your evaluation of their learning in particular lessons. This could contribute to UA3 and your Eportfolio (a key element of your evidence for the Teachers' Standards, especially standard 6).

1.5 ITE Stakeholders, Regulations, Policy and Practice

Initial Teacher Education (ITE) courses involves a range of stakeholders and are set within a number of regulatory frameworks, from the Department for Education (DfE), the Office for Standards in Education (Ofsted) and the Quality Assurance Agency (QAA) for Higher Education. These are just a few of organisations and stakeholders involved:

Government (DfE): Initial Teacher Training (ITT) - QTS

ITT criteria – e.g. specify minimum number of days in school

Core Content Framework – minimum curriculum entitlement, monitored as part of Ofsted inspections. (This is mirrored in your two ECT years – the same statements but without as much 'support and guidance from expert colleagues).

University: Initial Teacher Education – PGCE

Module and course specifications – QAA standards. External Examiner processes are an integral part of the quality assurance processes.

Schools: MATs, Teaching Schools

Curriculum choices – e.g. exam boards…

Students and Parents

Unions and Professional Bodies

Chartered College of Teaching – free for student teachers

Education and subject associations

e.g. ATM, MA – free for student teachers

BSRLM – reduced rate for students

Employers…

In terms of regulations, your course runs within government ITT criteria (for QTS) and university regulations for PGCE.

Chartered College of Teaching

Membership provides full access to Impact (professional magazine) and other resources.

Union Membership

These are usually free for student-teachers. (There is a range of 'political' positions taken by unions, including some with non-strike stances). They can be a good source of advice, independent of the university and schools and focussed on your needs – in the past, the vast majority of student-teachers have not needed to seek advice, but occasionally situations have arisen when support has been sought. You need to be a member beforehand, so my personal advice is to investigate the different organisations and join one before school practicums start.

1.6. School Practicums

1.6.1 'Paperwork': Eportfolio and School Files

Maintaining your Eportfolio and school file(s) is a critical part of a PGCE course. The Eportfolio is held on OneDrive; your School Files might be electronic, on paper or a combination of both.

Your Eportfolio contains key documents that allow you to plan for, enact and review your progress, whilst also allowing your tutor(s), co-tutors and other school-based staff to support and monitor your progress. Your Eportfolio will contain lesson plans, reviews (evaluations) and observation forms (ERFs) for formally observed lessons, plus any worksheets/resources used and any other lesson related documents.

Your School Files contains this for *all* your lessons, including background information for classes – this can be within your Eportfolio (My Stuff). All these need to be kept up-to-date and in good order (remembering they may contain students' personal information, so it needs to be kept 'secure'). For the duration of the course, these are 'semi-public' documents, in the sense that school and university staff will expect to be able to access them without notice.

Structure of your Eportfolio

See the course information.

Structure of your School File – Electronic or Paper

Overview: Any relevant information about the school and department.

Main Sections: Separate sections for each teaching group.

For each class, the following is needed:

Information (remember security of information):

- Names and student information
    - A seating plan (invaluable for learning names, possibly with pictures),
    - Background information, e.g. SEND
    - Attainment information
    - Assessment and homework marks and other such records.

For each class, planning and review documents:

- **Long term plan**: The relevant section of the departmental scheme of work
- **Medium term plan** or topic overview for each unit of work and related reviews
- **Lesson plans**, associated resources, and related reviews (evaluations).

1.6.2 Teaching Load

Full-time teachers usually teach 90% of lessons, with ECTs teaching about 80% of available lessons. However, form tutor periods and varying lesson lengths makes timetables an inexact science. Often, full-time teachers teach 22/25 lessons and ECTs 20/25. Whilst 25 lessons a week is a common pattern, schools do vary; we trust that you can convert the following to different totals if needed.

Phase A: You should have a teaching timetable equivalent to *about* $\frac{1}{2}$ of an ECT timetable (i.e. about 40% of available lessons).

i.e. about 10/25 lessons

Phase B: You should have a teaching timetable equivalent to $\frac{2}{3}$ of a ECT timetable (i.e. about 54% of available lessons).

i.e. about 13.5/25 lessons

However, this relates to lessons in which you are taking the lead in teaching a whole class. Classes where you are supporting the class teacher or where you are teaching small groups should not be included in this total.

It is clearly difficult to write a timetable with exactly $\frac{1}{2}$ or $\frac{2}{3}$ of a ECT timetable, and it is appropriate to have some team teaching / small group work, though this should be small in number. In order to produce a balanced timetable, you may, for example, in phase B take the lead teaching role in 13 lessons and then team-teach 2 lessons. There are additional activities, such as gaining experience in learning support or working with intervention groups, that can be included in your timetable. However, you will need time to plan and review your lessons so you should have at least 12/25 'free' lessons in phase A and 9/25 'free' in phase B to plan and mark work; you should be **on school premises** throughout the day (lunch may be negotiable). Remember, you also need to be prepared to stay in school for afterschool meetings.

Key Stages

In the government regulations that govern ITE courses (including PGCE), Key Stages relate to **age** and not the curriculum taught: **Year 9 are Key Stage 3,** even if they are taught a GCSE syllabus.
Year 7, 8 & 9 = Key Stage 3   Year 10 & 11 = Key Stage 4   Year 12 & 13 = Key Stage 5

For the 11-18 course, you should be **teaching** all the key stages that are taught in your placement school - this includes Key Stages 4 & 5 if these are present.

For the 11-16 course, you should be **teaching** both Key Stages 3 & 4 if present.

You may not get sole responsibility for an exam class or A-level groups, but may team-teach. However, you should be taking the lead in planning and teaching for a reasonable proportion of the time. (For example, if you are team-teaching two A-level lessons a week, you may lead in one lesson per week or 20 minute sections of each lesson. You may find that with these classes you start with a more supporting role, with the school gradually letting you take more of a lead as you demonstrate your subject knowledge and planning).

Preparation Week

The first week is listed as 'preparation' and activities will focus on observing lessons and planning for your teaching in week two. However, you are likely to see your classes at least three times a week. It is therefore appropriate, *if you feel confident and with the agreement of your co-tutor/mentor*, to start teaching some of the classes some of the time in the first week. In the past, some mathematics students have found it beneficial to either start teaching parts of lessons (e.g. starter activities) and/or pick up one class earlier than the second week. However, it is also perfectly normal to do no teaching in the first week, and pick up your classes over the course of the second week. Observation does not stop when you start teaching. Focussed observations are part of your weekly activities, but also continue to observe others to focus on issues you and your co-tutor have identified throughout your placement (this could include part lessons some of the time).

1.6.3 Your School and Departmental Involvement

You are expected to participate in school and departmental meetings and professional development. In addition, where possible, you should attend parent evenings, with your role negotiated with your co-tutor/mentor/ITE coordinator. (This may be observation, working alongside the class-teacher or taking a lead role under the class-teacher's supervision, depending on when the parents evening falls in the year).

School Induction and Wider School Experiences

When you arrive, you should have an induction that covers the **school's safeguarding policies and practices.**

Your ITE co-ordinator should provide opportunities, over time, for you to become familiar with wider school issues (see Role and Responsibilities section for details).

Practicalities

- Caffeine, office space, photocopying (how, how much, time scales), …
- Access - keys (fobs), IT access, toilets
- School and departmental policies and practices
  - Behaviour policy (both policy and practice),
    - Including lending students equipment in lessons, student dress code and letting students out of class for toilet breaks etc.
  - Assessment policies (e.g. target setting, topic tests).
  - Groups of students e.g. SEND, Pupil Premium (though unless there is a specific issue related to learning you may not be entitled to this information)
- Staff dress code (aim for the smart end of what you see in school)
- Key staff; introduce yourself *when appropriate:*
  - Your co-tutor/mentor, departmental colleagues, heads of year, reprographics team (or the photocopier), cleaners, IT technicians, receptionists, ITE coordinator, head teacher…
- ITE coordinator expectations w.r.t. wider school involvement e.g. pastoral responsibilities, break duties, meetings you should/could attend, parents' evenings, extra-curricular activities. In general, take advantage of opportunities to extend your experience and get involved in school life, but do not over-extend yourself.
- Your timetable (but do not worry if this takes a few days)
  - Including times (esp. if split lunch), dates if a two-week timetable and rooms
- Lesson expectations
  - Format of lesson plans, when/how these need to be 'handed in' and to whom?
  - Registers in lessons; who and how?
- Departmental resources,
  - SoW and resources (where, what, access…)

- Classroom resources (e.g. exercise books, pens, visualizers, whiteboards…)
- Homework: when and how is it set, what type, expectations w.r.t. marking and any non-completion?

1.6.4 Teaching practice

By the beginning of each full-time practicum, your co-tutor/mentor should provide your teaching timetable that meets the requirements set out above. This should include a range of attainment if the classes are organised into sets.

Fill in the Eportfolio timetable as soon as possible – you do not need to wait until you have all the details – fill in what elements you can as soon as you can**.** When possible add times that would be suitable for a UoL tutor to visit and undertake a join observation with your co-tutor/mentor. When this visit is undertaken, both you and your co-tutor/mentor need to be available after the lesson for *at least* 30 minutes (usually 40-60 minutes) – for School Direct students this might be undertaken by the SD Lead.

Feedback

- Once teaching, you should expect at least one formal observation a week, with verbal and written feedback summarised on an Evidence Record Form (ERF). There will also be many less formal observations. Your co-tutor/mentor should conduct some of these observations, but other experienced teachers can undertake both formal and informal observations.

- There should be a weekly meeting with your co-tutor/mentor to discuss planning and progress, to talk over difficulties, set targets, etc. This should be in a timetabled slot in the week. You should also see the ITE co-ordinator regularly, usually with the other student-teachers at the school.

- Your Eportfolio and school file(s) should be available for your co-tutor/mentor, ITE co-ordinator and UoL tutor to see at any time – you need to share the Eportfolio (OneDrive) with write access (default is read only so change to write access before sharing).

Lesson Planning: Placement Requirements

Read Section 3: Curriculum Planning for the details of the planning process. You should plan and review (evaluate) each lesson that you teach; however, how you undertake these activities will evolve over time.

- Your co-tutor/mentor will oversee your planning. They, or the regular class teacher, may want to see individual lesson plans or, as practicums progress, they may be happy to review your medium-term plans. You will usually be asked to share your planning a set period ahead of the lesson (e.g. 48 hours) so there is time for your co-tutor/mentor or class teacher to comment in case you need to make any changes. One efficient way would be to use your Eportfolio as both you and your co-tutor/mentor has write access.

  - ♦ **Golden Rule:** You **must compete all tasks** that you are going to give to students.

    If you do not do this, we support the school taking you off teaching that class.

    (For key questions/elements this should be done in a variety of ways, using different representations, identifying barriers and common misconceptions).

- Try to outline your medium-term plan *before* planning individual lessons – more efficient and produces more coherent lesson sequences (albeit difficult to begin with). Also, your co-tutor/mentor/class teachers may then be more flexible w.r.t. individual lesson plans.

UoL Tutor Visit

- If your UoL tutor visits they usually undertake a joint lesson observation with your co-tutor/mentor. (The whole lesson may be observed or the UoL and co-tutor/mentor may watch the majority of the lesson but leave for a short period). Afterwards the UoL tutor will join you and your co-tutor/mentor for your post-lesson debrief. Please have your Eportfolio and any school file available. Although UoL tutors are really interested in seeing you teach, and will contribute to the feedback, their role is to moderate the assessment of you by your co-tutor/mentor/the school. If School Direct, this role may be undertaken by the SD Lead.

    ♦ The majority of the professional discussions will involve all parties (co-tutor/mentor, UoL tutor and yourself). Whilst it is common for the UoL tutor to speak to the student-teacher on their own for a short time, please **ask** if you want this to happen, either when the UoL tutor visits or via email beforehand if this is more appropriate.

- If there are any issues then please contact your UoL tutor (core) or your lead school (SD) as appropriate in the first instance.

Ways of Working:

It is not automatically true that the more you teach, the better you get. Time and effort spent reading, thinking, reflecting, observing others and working with students in various ways will enhance your own classroom performance. Consider:

Teamwork:

- Collaborative planning, teaching and evaluation
- Undertake different roles: work with small groups, students with SEND
- Be creative: try new ways to use software, design activities, audio/video record (with permissions in place), work with other departments…

Structured observation:

- See 'Observation of others ⇔ Teaching' section for details
- Observe other subjects

Studying and using systems for assessing, recording and reporting progress:

- Find out how teachers assess their students and keep records

Parental contact

- Under teachers' guidance contact parents as appropriate

How well do you know your school?

During the first few weeks in your partnership school, you can review your understanding of the school context by considering the following questions.

1. Norms of expected behaviour

What routines are taken for granted; how did teachers establish these?

What are the norms of behaviour in the classroom?

Are there differences in expectations between staff, if so what and why?

2. Gaining and maintaining attention

How do teachers gain the attention of the class?

How does the teacher deal with noise, disruptive behaviour and lack of attention?

How many changes of activity are there in a typical lesson and how are transitions made?

How do teachers engage students in a plenary when students may simple pack up early?

3. Using resources

How do teachers *and students* use technology, visual aids, textbooks and paper resources?

4. Classroom organisation

How do teachers ensure that students understand what to do and the purpose?

How often do students work as a class, individually, in pairs, in small groups (and what principles are used for organising students)?

What different ways of working are used (e.g. discussion, practical activity, mathematical software on handheld technology)?

What provisions are made for students' different work-rates?

5. Marking students' work and assessment

What system do teachers use for coping with their marking load?

What criteria are used for marking students' work (how is poor work followed up)?

How is feedback given to students and to parents?

What different modes of assessment are used in the department?

6. Studying Schemes of Work

How is the development of Schemes of Work undertaken in the department; for example, is there collaborative planning, shared resources, the use of an overarching approach such as ‘mastery’?

How did the department decide which examination syllabus to use for GCSE or A Level?

1.7 Roles and Responsibilities in Partnership

1.7.1 Student-teacher

In the early part of the course, there seems to be a lot to do and initially you may find it difficult to picture how it all fits together. As the course progresses, everything will (hopefully) fall into place. All the tutors with whom you work will assist and prompt you, but you are expected to take responsibility and take proactive action in a way befitting the profession of teaching - it is well worth spending time early on in getting organised.

You will need to:

- To monitor your progress against our curriculum (CARDs), benchmarked against the CCF and ultimately the Teachers' Standards. Whilst on school practicums your weekly meetings are central to this process, so plan and take appropriate action including:
    - ♦ Set realistic, achievable targets with all your co-tutor/mentors' and tutors' help.
    - ♦ Plan, 'experiment', reflect, review your teaching, and thereby improve.
    - ♦ Deepen your mathematical understanding, both subject knowledge and pedagogical content knowledge – tracking your progress on the subject audit.
    - ♦ Maintain and engage with course processes, such as your student reflections, focussed observations, subject knowledge audit and progress documentation (e.g. weekly meetings)
- To engage with educational research, developing your critical reading and writing in the social sciences, drawing on UoL (University of Leicester) support services as appropriate.
- Conduct yourself in a professional manner, including: the development of good working relationships with school and university staff, fellow students, and students and their parents; and professional practices such as notification of absence, meeting deadlines and honouring commitments.

1.7.2 University Tutors

The tutors collaborate to coordinate and support your experience through the year and to ensure a high quality, practically-based University seminar programme. Their role is to:

- Plan and teach the mathematics component of the PGCE
- For student-teachers on the University-led course: Coordinate and organise your Phase A and B placements
- To visit and observe your teaching when in school
- To moderate the schools' judgements
- To set and assess your university assignments
- To oversee your work as a whole, drawing on all evidence to assemble a comprehensive reference
- To offer support and guidance to ensure continuity and progression towards Qualified Teacher Status (QTS)

1.7.3 Subject co-tutor/mentor (Partnership School/College)

Your subject co-tutor/mentor: The person in school with a designated responsibility for your development as a mathematics teacher and for assessing and reporting on your progress in school.

Subject co-tutor/mentors' role:

- To induct you into the department and ensure you are familiar with all appropriate departmental resources and course requirements (e.g. syllabuses, schemes of work, assessment systems)
- To establish an appropriate timetable, with opportunities to observe, teach and become involved in the life of the school/college
    - Note: Key Stages are defined by age not syllabi, so **Year 9 is Key Stage 3.**
- Supervise and mentor you through regular tutorials, setting targets and strategies as appropriate
    - ♦ You should have one co-tutor/mentor meeting a week, which ideally is set for the same time each week when both you and your co-tutor/mentor is 'free'. Discussions to include:
        - A review of your progress, including scrutiny of your documentation
        - To shape opportunities for you to make progress towards the Teachers' Standards by drawing on our curriculum and the CCF framework
- To write and submit interim and final reports on your progress against the CARD and ultimately the Teachers' Standards

1.7.4 Co-ordinator for Initial Teacher Education (Partnership School/College)

The ITE/ITT Coordinator in each partnership school is a senior member of staff who assumes overall responsibility for your work in school, with particular responsibility (in conjunction with the University) for your progress in Standards Area 1: Professional Values and Practice.

They should also provide you with an overall induction to wider school issues; including:

- Safeguarding: policy and practice as related to their school
- School ethos and aims
- SEND provision
- Data management
    - ♦ E.g. Progress 8 and Attainment 8
- Assessment Policies and Practices
    - ♦ E.g. marking policy
- School wide support programmes for student progress, including student premium
- Workload policies and practices
    - ♦ Including school planning and marking policies
- Pastoral systems
- CPD provision

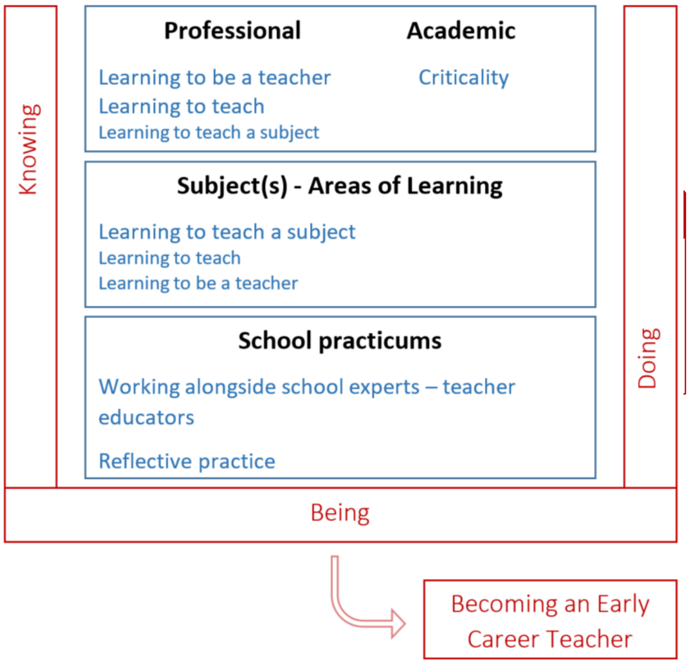
2. Teaching and Learning: Complex and Multifaceted

2.1 Knowing, Doing, Being: Programme Structure

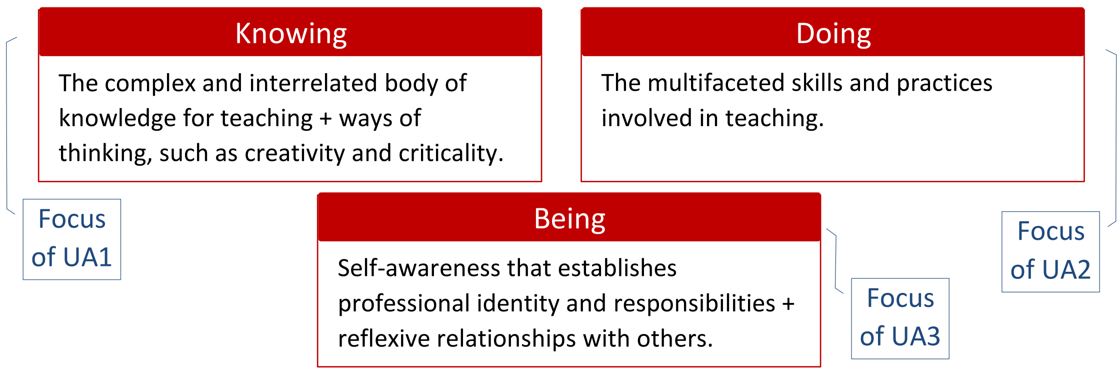
Learning to Teach a Subject, namely mathematics, is the key focus of the mathematics course. However, this cannot be separated from other elements of becoming a teacher.

University days: The Professional and Academic component will tend to focus on Learning to be a Teacher and Learning to Teach, with the Subject component tending to focus on Learning to Teach Mathematics, but these all interlink.

School Practicums: You will work with and alongside school experts (you co-tutor/mentor and other teachers), where you will develop your practice.



The model of knowing, doing, being (becoming) (Craig, 2018) is a way to articulate the complex process of integrating theory and practice, thereby shaping your own persona as a teacher of mathematics/children.



A simple example is to think of learning to drive. We can know the ‘rules of the road’ (knowing) but we still need to get behind the wheel to learn to drive (doing), which requires practice. To be a responsible driver, amongst other things, we need to understand our skill level whilst also considering other road users (being), where reflection could improve our driving.

2.11 Our Curriculum: CARD

The PGCE curriculum is **summarised** in the Curriculum and Assessment Review Documents (CARDs) – one for each phase. You will see CARD references in taught sessions and it is used for the assessment of your school practicums. These are 'dense' documents – do not expect to instantly understand everything – overtime, revisiting the documents, you will develop the required understanding.

Embedded within our curriculum are two DfE requirements, the Teachers' Standards and the Core Content Framework (CCF).

You will be assessed against the Teachers' Standards at the end of the course – the explicit benchmark statements are contained in CARD B. If you are 'on-track' in phase A you will be making appropriate progress towards these Teachers' Standards.

The CCF is the minimum curriculum entitlement that is set for all Initial Teacher Education courses by the DfE (and the same statements form the Early Career Framework for your two ECT years). This minimum is embedded within our wider curriculum. To support your understanding of the CCF and ECT explicit reference will be made to particular CCF statements when appropriate and directed activities will allow you to interrogate this framework.

Our Curriculum	CCF Priorities	CCF term	DfE Teachers' Standards and statements
A) Academic: Postgraduate Study	*Links to all five other areas e.g. through developing knowing, doing, being through engagement with university assignments.*		
B) Professional Behaviours and Values	Professional behaviours CCF5	Professional Behaviours	S8 Fulfil wider professional responsibilities PPC Personal and professional conduct (Part 2)
C) Behaviour and Relationships	Behaviour management CCF1	High Expectations Managing Behaviour	S7 Manage behaviour effectively S1 Set high expectations
D) Pedagogy	Pedagogy CCF2	How Pupils Learn Classroom Practice Adaptive Teaching	S2 Promote good progress S4 Plan and teach well-structured Lessons S5 Adapt teaching
E) Curriculum	Curriculum CCF3	Subject and Curriculum	S3 Demonstrate good subject and curriculum knowledge
F) Assessment	Assessment CCF4	Assessment	S6 Make accurate and productive use of assessment

The mathematics specific content tends to relate to (D) Pedagogy, (E) Curriculum and (F) Assessment, and there are subject exemplar statements on these sections of the CARDs. However, the interconnected nature of teaching means that none of these areas can be seen in isolation.

2.12 Core Content Framework

The Core Content Framework (CCF), your minimum curriculum entitlement, has two threads, namely: '**Learn that…**' and '**Learn how to…**'. We see these as part of **knowing** and **doing** respectively, but with the caveat that there is a reflexive relationship between these threads, drawn together through your reflective practice as you develop your professional identity (being).
The CCF does not have any subject specific content, so it is important to consider what the statements 'look like' through a subject lens. Some examples have been selected below to illustrate where they fit within the mathematics strand (please see complete CCF for all the statements).

When planning lessons, modelling, questioning, classroom talk all play a vital role – these are all covered in detail in section 3.

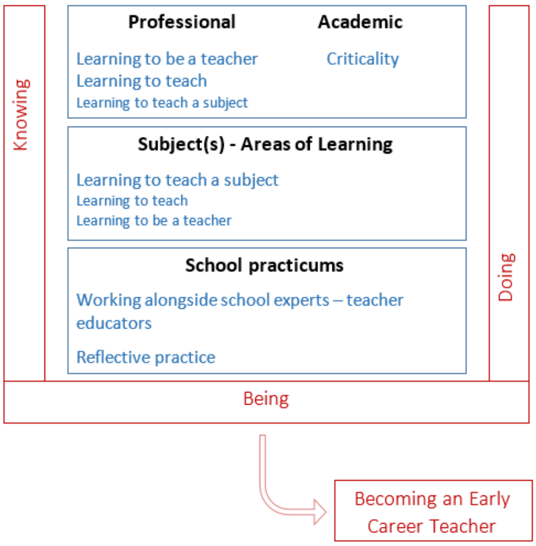
(D) Pedagogy: Classroom Practice (S4)

Learn that…
4.3 Modelling helps pupils understand new processes and ideas; good models make abstract ideas concrete and accessible.
4.6 Questioning is an essential tool for teachers; questions can be used for many purposes, including to check pupils' prior knowledge, assess understanding
4.7 High-quality classroom talk can support pupils to articulate key ideas, consolidate understanding and extend their vocabulary.
Learn how to…
4B Make good use of expositions, by:
4f Discussing and analysing with expert colleagues how to use concrete representation of abstract ideas (e.g. making use of analogies, metaphors, examples and non-examples).
4h Combining a verbal explanation with a relevant graphical representation of the same concept or process, where appropriate.
4C Model effectively, by:
4j Narrating thought processes when modelling to make explicit how experts think (e.g. asking questions aloud that pupils should consider when working independently and drawing pupils' attention to links with prior knowledge).

Setting is a particular issue for mathematics – see section 2.6.

4.10 How pupils are grouped is also important; care should be taken to monitor the impact of groupings on pupil attainment, behaviour and motivation.

Learn that…
E.g. 3.8 "Students are likely to struggle to transfer what has been learnt in one discipline to a new or unfamiliar context"



Learn how to…
E.g. "Support students to build increasingly complex mental models, by: Discussing and analysing with expert colleagues how to … teach key concepts through a range of examples."

2.2 Mathematics and Its Learning

Mathematics is fascinatingly complex, but its cohesive structure brings huge benefits when you grasp a concept. As then you are able to 'compress' elements that were previously held separately into a compact notion (e.g. Tall, 1995). This means you can move onto more complex concepts, using these earlier ideas as building blocks. For example, it takes a long time to learn to count, but once you have understood place value and addition you can apply this to any positive integers, then any integers, then rational numbers, then all real numbers, then imaginary numbers… (remember this when meeting Cognitive Load Theory!)

You are successful learners and successful learners of mathematics. Your progression means that, in relation to school mathematics, (i) some ideas are so well used they are part of your 'intuition' – i.e. you draw on these ideas/processes automatically (Watson, 2019b) (ii) you have the big picture (Sfard, 1991).

> An analogy: When meeting novel concepts, it is like walking into a blacked-out room. You learn by wandering around, bumping into things, gradually building up a picture of the room, but it may be far from complete. If/when you find the light switch all becomes clear and 'neat' – you can see how it fits together; it is impossible to go back to your earlier understanding of the room (you have found the light switch - it is hard to put yourself in the place of the learner who is still in the blacked-out room…).

To teach, a first step is to 'unpack' your own thinking about mathematics, so you can articulate these ideas more explicitly to yourself, your peers and to learners.

Moreover, it can be easy to describe how to 'do the mathematics' (skills to apply procedures, algorithms, rules to some examples) – it is much harder to articulate the 'why' (understanding the mathematical structures, concepts, generalities…). The latter is needed to successfully induct learners into mathematical learning.

The unpacking: **Do mathematics, think about mathematics, talk about mathematics**.

Some good mathematical activities with articulated rationales:

> Section 4: Algebra Equivalence Task
> The ICCAMS Resources - Blackboard
> The Standards Unit - Blackboard

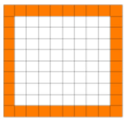
Another complexity when thinking about learning is that mathematical concepts are multifaceted, and cannot be fully captured with a single example or representation. An expert, looking back on a concept that they have used many times, are likely to have this 'neat', coherent and compact version. If an expert deconstructs this concept into steps for the learner, the temptation is to think that the learner can reconstruct the concept with equal clarity by following those steps… this is not the case* (Confrey et al., 2014). Learners need to meet these concepts from a range of perspectives, and they themselves, have to work on how these perspectives 'fit' together; in this respect learning is 'messy'- so give students the opportunity to play with ideas.

> *An example: Remember proofs? Many mathematics undergraduate students find proofs hard to understand (Knuth et al., 2019). Whilst the mathematicians that developed these proofs will no doubt have spent many hours with calculations, diagrams, changes of direction, trying with special cases, backtracking from dead ends, … the proof is presented in its final neat version. The hours spent 'playing' with the

> mathematics in generating the proof means those mathematicians have that rounded perspective that enable them to both understand the proof and appreciate the final succinct presentation. For learners, without that opportunity to spend time exploring the concept from different perspectives, they often find this neat presentation difficult to access, no matter how clearly the steps in the proof are explained.

In other words, to learn, time is needed to engage with ideas from different perspectives, revisiting ideas, testing out understanding, extending the range of examples met and representations used… Effective recall is more than recalling individual 'facts', it is about how individual pieces fit together and how ideas relate to different examples, problems and contexts.

Whilst mathematics may be a coherent body of knowledge for some, it has been shown that we can hold mathematical ideas that would be mutually incompatible from an expert mathematician's perspective (Küchemann and Hoyles, 2005).



For example, how problems are presented can 'cue' different types of recall. A n^{th} term tile problem presented as a diagram might generate $4n - 4$ or $4(n - 1)$ or… but a table of values (4, 8, 12, …) might cue $n + 4$. An expert would recognise these expressions should be equivalent, but novices are far less likely to notice. For many learners, unless cognitive effort has been spent to link ideas together, mathematics can remain a collection of isolated 'facts' to be remembered (Skemp, 1976), allowing these contradictions to remain. It is our job as teachers to orchestrate this connecting, as it is unlike to happen spontaneously for many.

2.2.1 Secondary Mathematics PGCE Curriculum

The attempt to articulate the 'knowing' part of the Secondary Mathematics PGCE curriculum has led to a 60+ page handbook. So, what follows is an overview of elements of taught sessions, with some issues unpacked further in the following sections. Some elements are addressed directly in particular taught sessions, whilst other aspects are integrated across a number of sessions; the former will be more easily identified below, with the latter emerging as the course progresses. The gold represents how the curriculum has been shared with co-tutors/mentors in school in summary form.

Pedagogical Content Knowledge is the focus of taught sessions, rather than Mathematics Subject Knowledge. To 'get at' the mathematics and its learning, the aim is for students to:

Do mathematics, think about mathematics, talk about mathematics.

With a limited amount of time, the aim is to provide ways of thinking about mathematics and its learning with exemplars that student teachers can apply to other topics and contexts which they meet whilst on school practicums. Whilst subject knowledge will develop during the course, the student teachers should bring sufficient subject knowledge with them to undertake self-directed review and development during the course.

Whilst they interrelate, the curriculum is articulated in relation to three mathematics foci, plus academic reading and writing:

- The learning of mathematics: Theories of learning e.g. variation theory.
- The teaching of mathematics: Pedagogy
- Topic specific issues: Number, algebra, proportional reasoning, geometry, statistics
- Academic reading and writing: UA1, UA2 & UA3

Phase A University Sessions: This focusses on introducing key ideas and examples through which to explore these key ideas. And with sufficient rehearsal of lesson planning to produce sound lesson plans drawing on school resources and cotutor support.

Phase A School Practicum: The focus is on selecting appropriate tasks and examples, mapping out precise and concise explanations and monitoring student responses.

Phase B University Sessions: This focusses on revisiting key ideas and their relationship to their phase A practicum experiences, with a view to deepen knowledge and to prepare for their second school practicum. Rehearsal of planning now focusses on developing flexibility to respond to student need.

Phase B School Practicum: The focus now shifts towards student engagement and ownership of the mathematics.

	Topic Areas
Topic: Number	Operations: Relationships; order of operations (commutative, associative, distributive) Different models for operations, cued by different pictorial representations (Phase A x). Early number: Counting principles (1-1, order irrelevance, fixed order, abstraction, cardinality) as example of abstraction and compression (APOS); Additive relationships: the range of structures (aggregation, partitioning, augmentation, reduction with + -); different levels of difficulty dependant on problem structure; Introduce non-statutory guidance (KS1,2 &3) and ‘promoted’ approaches therein (transition) Equivalence: Relational vs operational understanding of the = sign. Fractions: Sub-constructs (part-whole, ratio, quotient, magnitude, operator); representations; common misconceptions, including natural number bias.
Topic: Algebra	Letters: The different roles letters can take (Logic of the discipline: letter as specific unknown, generalised number and variable. Common misconceptions: letter evaluated, not used, or as object) Links to prior knowledge: Esp. How number relationships are introduced and use of letters in science. Representations: Different representations, their affordances and switching between representations, inc. between process (e.g. table of values) and objects (e.g. graphs).
Topic: Proportional Reasoning	Concepts: What does ratio mean, what does proportion mean? Key concepts and misconceptions: Multiplicative reasoning; the unit whole; constant of proportionality; two relationships ‘scalar’ & ‘functional’ Complexities: Identifying multiplicative relationships, including what ‘provides’ the constant of proportionality. Multiple soln. strategies (e.g. two relationships), for some (nice) number additive approaches work Representations: Different representations, their affordances and critical features, including double number lines. Contexts: Enlargement (circles) to explore why x rather than +; Shadows, elastic bands… Planning: Exploration of ICCAMs resources that include pedagogical reasoning Pedagogy: Take care not to use pseudo problems (e.g. time if run twice as far…)
Topic: Statistics and Probability	Probability concepts: Randomness, sample space, outcomes, inc. equally likely contexts, Types: experimental, theoretical, historical; discrete & continuous distributions, Normative thinking: Law of large numbers (local randomness, global predictability); independence, relative frequency Common misconceptions: Outcome orientation, representative heuristic, gambler’s fallacy, equiprobability bias Learning: Can hold contradictory notions, cued by (marginally) different contexts. Correct responses ≠ normative thinking. Reverting back to previous thinking is common Attribute framing: Level of numeracy affects the level of influence of the ‘frame’. Ratio bias. Statistics Learning procedures can inhibit understanding of averages and spread Sample Population debate. Inferential statistics: informal and formal Common misconception: Correlation fallacy
Topic: Geometry	Learning: Seeing General ⇆ Specific (e.g. static diagram representing a general or specific case). Attention: Ignoring and extending diagrams as an example of noticing and shifting attention. Dynamic geometry: Role in proof - picturing ‘all’ cases. Concepts (and misconceptions): Angles: Misconception of angle – e.g. size thought of in relation to line size, measuring issues E.g. Overuse of standard orientations can lead to non-recognition.

2.2.2 Secondary Mathematics PGCE Curriculum Summary

Do mathematics, think about mathematics, talk about mathematics.

The learning of mathematics

The role of examples
Details and sequencing matters.
Example space: Extend - typical, atypical (e.g. non-standard orientation) boundary, *not* examples.

Variation theory: Critical aspects – increasing visibility; what is varied, and how, sequencing.

Process-Object: e.g. $\frac{2}{3}$ as $2 \div 3$ or a point on a number line/number that can be manipulated.

Multiple Representations: Identifying critical features, the links between representations and to concepts; their power and their challenges.

Cognitive Science: E.g. a mathematics perspective on cognitive load (e.g. Ollerton et al., 2021).
Mathematics 'contains' many feature to help

The teaching of mathematics

Classroom interactions
Questioning: Broad range of question stems; developing IRE sequences; complexity of interpreting responses in terms of reasoning.
Balancing exploring student reasoning vs 'mathematical horizon'

Scaffolding: e.g. direction maintenance; marking critical features; reducing degrees of freedom, with clarity about what is 'left' for students.

Noticing, attention, explicit-implicit; arbitrary-necessary

Planning: Articulate mathematical concepts.
All mathematics completed – linked to concepts.
'Standard resources' thought about and 'tweaked' – marking structures/concepts
'I do, we do, you do' *plus interrogation*
Plan examples, models, board work, explanations in detail, anticipate student responses and yours.

Curriculum
E.g. KS3 Non-statutory advice, ICCAMS

Number

Operations: Representations; relationships; order of operations (commutative, associative, distributive)
Equivalence: Relational vs Operational understanding =
Fractions: Sub-constructs (part-whole, ratio, quotient, magnitude, operator); representations; common misconceptions, including natural number bias.

Algebra

Letters: The different roles letters can take
Links to number
Switching between representations, including switching between process (e.g. table of values) and objects (e.g. graphs)

Proportional reasoning

Multiplicative Reasoning
Key concepts (and misconceptions): Identification of multiplicative & proportional relationships; unit whole; constant of proportionality.
Scalar and functional multipliers.
Representations, including double number lines. e.g. through engagement with ICCAMS materials

Geometry

General⇔ Specific: Shifting between a static diagram representing a general case or a specific example.
Attention: Ignoring and extending diagrams.
Dynamic geometry: Role in picturing 'all' cases.
Concepts (and misconceptions): E.g. angles; standard orientations.

Probability

Concepts (and misconceptions): Law of larger numbers; local randomness, global predictability; independence (outcome orientation, representative heuristic, equiprobabilty bias)

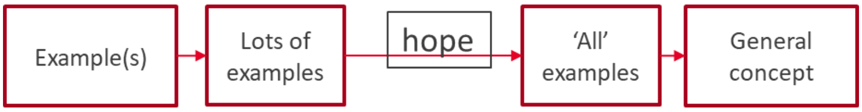
Statistics

Calculation of measures of central tendency/dispersion vs understanding what these data mean.
Population vs sample situations.

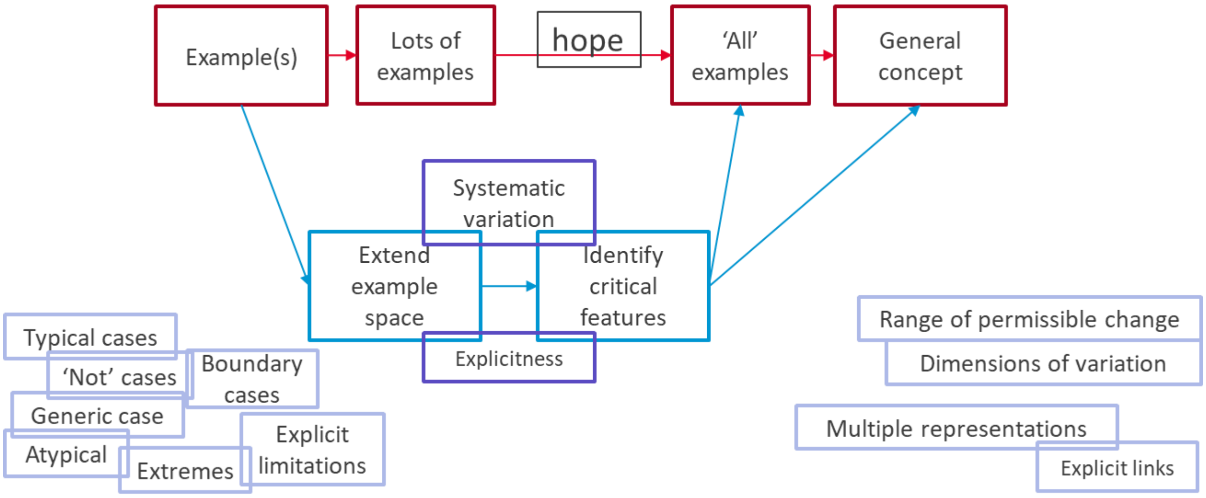
Key perspectives

Developing the use of examples:

A) Approaches sometimes adopted by student teachers

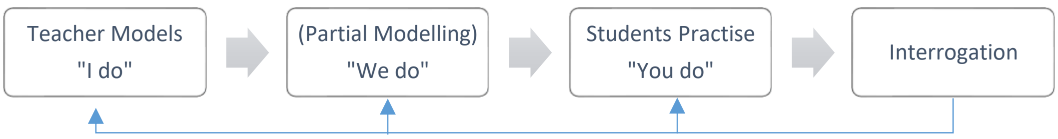


B) Closer attention to examples used, and how those examples can be interrogated, provide more structured routes to generalisations.



Planning for interrogation

In phase A, student teachers will be looking to develop the foundations of effective planning. The modelling of examples followed by pupil practice is likely feature regularly. They are encouraged to select 'standard resources', which they think hard about – finding ways to draw attention to key mathematical features/concepts and ways to 'tweak' the task to support and/or extend understanding. This interrogation of the work completed can happen at multiple stages of the task.



Tweaks and questioning stems:

Exemplifying: Is this an example of… What makes this an example of…
Changing: What needs to be changed so that… What would happen if…
Comparing: What is the same and different about…
Varying approach: Find alternative strategies, compare – which is more reliable, efficient…
Representations: Draw a diagram to represent this problem – what problem could this diagram represent. How is this… represented in the diagram. What are the key (critical) features of the diagram …
Generalising: Describe all the possibilities, how do you know. What can change/ what has to stay the same so… is still true? What happens in general?
Explaining: Give a reason for using/not using (self-explanation), how can we be sure…

2.3 The Role of Mathematics Education Research – Your Knowing

Wider reading is an essential element in developing your understanding of teaching and learning of mathematics.

Mathematics education research is a vast field, covering topic specific issues (e.g. the use of the equals sign), to the more general features (e.g. the role of multiple representations, group work, mathematical discourse and issues related to social justice). There are also mathematics specific theories of learning, such as variation or process-object theories and Piaget's logico-mathematical knowledge. In order to develop your understanding of the classroom in an effective way, you should be drawing on these theoretical perspectives in your planning, teaching and review activities. Starting points could be:

> Nunes, T., Bryant, T. and Watson, A. (2009) *Key understandings in mathematics learning*. Nuffield Foundation.
>
> Watson, Anne, Jones, Keith and Pratt, Dave (2013) *Key ideas in teaching mathematics: research-based guidance for ages 9-19*. Oxford: Oxford University Press.

See the Mathematics Blackboard reading list for access.

Maintaining a critical* stance is an essential part of developing an understanding of assumptions and presumed knowledge (both your own and, with care, those of others). This stance enables you to challenge these assumptions, where appropriate, as well as consider the relevance to your context and compatibility/coherence/contradiction with other perspectives. Engagement with education research and periodicals provides a framework for developing your criticality*.

> * Criticality - meaning: 'Involving the objective analysis and evaluation of an issue in order to form a judgement'. **Not: '**Expressing adverse or disapproving comments or judgements' (Oxford, 2019, critical entry).

2.3.1 Learning mathematics

Learning is not visible…

Learning is a complex 'hidden' construct – we cannot see it directly. We need models and theoretical frameworks to interpret observable behaviours. One challenge is that our own ideas (models/theories) of learning may be held implicitly rather than explicitly, so identifying and articulating how we interpret our observations can be obscured (Cobb et al., 2011). It is argued the same is true for a significant part of teacher expertise, with expertise held as heuristic models build up from experience, rather than well-defined, explicitly articulated concepts (Watson, 2019b). This can make this expertise more difficult to access, articulate, share with other and develop.

And what is mathematics: Representations and abstraction

Some argue mathematics is unique due to its abstract nature (e.g. Sfard, 1991) – with concepts represented through different representations and forms.

For example, a single quadratic equation can be represented algebraically (in different forms), graphically, diagrammatically and with a table of values. Quadratics (all quadratics), as a notion, has particular critical features, but no one single 'touchable' representation captures the full nature of the concept (Dreher and Kuntze, 2015). A fuller understanding would involve: the valid region/the field of play (any real numbers, Cartesian plane…); the different representations and their affordances, links and transformations between representations; the notion of equivalence, solving; the role of letters...

Another common example is fractions, with a range of sub-constructs and representations

Sub-constructs: e.g. $\frac{3}{4}$	Part–whole	three out of four equal parts,
	Ratio	three parts to four parts,
	Quotient	three divided by four,
	Measure (magnitude)	a point on a number line,
	Operator	three quarters of a quantity diagram

There are a great range of representations (e.g. subdivided shapes, number lines…), some of which are more likely to 'cue' some sub-constructs than others (Rau and Matthews, 2017).

Teaching for Understanding: The why?

Why 'the why' is hard

- No single representation or example can fully capture a mathematical concept
  - Concepts have many (multiple) representations
  - Understanding generality often arises from specific examples
    - Experience of 'variation' is needed to gain a more complete understanding of the concept (e.g. Watson and Mason, 2005)
- Learner's understanding can remain unstable for a long time
  - The mind can tolerate inconsistencies (often not even recognised)
  - We can all 'flit' between different ways of thinking
- Experts can apply flexible thinking (as they hold all the cards to choose from), selecting, often intuitively, the most effective route – hard to teach to novices holding few cards
- Takes time…

2.3.2 Implicit/Explicit and Noticing – what we attend to

Heuristics (Intuition) and Consciously Deliberate Acts

The world is complex. To cope, through our experiences, we build up heuristics – mental 'short cuts' to help with efficient decision making in situations with which we are familiar* (Watson, 2019b) (*easing cognitive load – by the way this is a theoretical perspective…).

> For example, once we are proficient at driving, decisions about slowing down for traffic lights, changing gear etc. do not need to be taken at a 'consciously deliberative' level. This allows us to focus on other issues, such as pedestrians.

In social situations, we build up a shared understanding of normative behaviours (Cobb et al., 2011). For example, depending on the cultural context, a certain level of eye contact is the norm. Actions and responses in these typical situations can often be taken at a heuristic level – much of the time, we do not make deliberate decisions about eye contact. In these types of circumstances, we tend to notice the unusual rather than norms, which includes our behaviours as well as that of others – we notice staring or no eye contact. One issue for teachers is that normative behaviours are very important in classrooms, and we may not notice. For example, we may have developed a mixture of 'good and bad' habits, such as only asking 'why' when we treat a student response as an error, or converting a partial student response into a complete mathematical statement ourselves rather than ensuring the student clarifies/completes. And we need to notice to change…

Classroom Norms

Classrooms are complex, dynamic places, with teachers having to make multiple in-the-moment decisions. Professional expertise builds up heuristics to facilitate decision-making; useful in allowing norms to be managed without too much cognitive effort, so the teacher's attention can be 'spent' elsewhere (Watson, 2019b). But this also adds complexity: (i) these heuristics are not available to those new to the profession (ii) these decisions are more difficult to access, say by a co-tutor/mentor sharing expertise with a student-teacher (iii) these decisions may not be noticed, making it more difficult to understand their impact on practice and thereby more difficult to build on or change, if appropriate.

Classroom norms offer a window into implicitly held views of what is 'taken as shared' in mathematics classrooms.

For example, a teacher might control most classroom interactions through IRE exchanges (a common occurrence in UK mathematics classrooms):

Initiate - a question asked by the teachers with one expected solution
Response - by a student
Evaluate - by the teacher
Often: 'correct' is acknowledged straight away
'incorrect' prompts a follow-up by the teacher (e.g. asking why, asking a simpler question, asking another student)

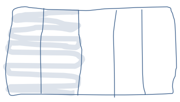
You will need to understand the potential impact of normative patterns of interaction. For example, **if** IRE is the dominant form of interaction, then the 'taken as shared' view of mathematics in that class is *likely to be that*:

- If single solution questions dominate: *taken as shared view* - mathematics is about finding *the* answer.
- If correct response = immediate accept: *taken as shared view* - mathematics is about find the correct answer as efficiently as possible.
- If incorrect response = follow-up: *taken as shared view* - there is an 'error' and 'errors' are to be avoided.
- If the teacher always evaluates: *taken as shared view* - the teacher is the arbiter of correctness and is responsible for student understanding.

Teachers might say that errors are part of learning and that different opinions are valued, but norms of behaviour built up over time outweigh a few verbal comments. If the above types of interactions are your pedagogical norms, they are less likely to be deliberate conscious actions, and you are less likely to notice their impact – you may believe you are producing a positive error climate in your classroom but your own normative behaviours could be communicating a very different message, namely that errors are to be avoided – all of which you may be unaware.

Noticing

What is noticed varies between people, influenced by our knowledge, beliefs and values (Mason, 2011b). We do not see all – and experts usually 'see' differently from novices – experts usually have an intuitive, hopefully better, understanding of what to pay attention to and what to ignore (Van Es and Sherin, 2002).



E.g. We (experts) would probably interpret this as representing $\frac{2}{5}$, assuming the larger rectangle represented a unit whole, ignoring the fact that the sections are not actually equal, and also understanding that different shapes and sizes could be used and the colour has no mathematical relevance (Rau and Matthews, 2017). Novices are less likely to see (recognise) the mathematically salient features and are more likely to pay attention to (be 'distracted by') other elements.

Implicit/Explicit

There can be a tendency for key mathematical ideas to be communicated implicitly rather than explicitly (Mason, 2011a). For example, attention may not explicitly be drawn to the mathematically **critical features** (Watson, 2017). In the fractions case, these critical features would include: the notion of the unit whole, with the corollary that the rectangle could be any size, and indeed swapped for any shape as long as there was equal partitioning of area forming the part-whole representation.

Often examples are used in mathematics to illustrate more general concepts (see 2.3.4). The notion of an **example space** is the range of examples a person associates with a concept (Watson and Mason, 2002). For example, once common misconception related to multiplication is that it makes things bigger. This can arise when the examples students are exposed to is limited to positive integers, where products are bigger than the multiplicand and multiplier. So, we should look to extend students' example space over time – in this case use numbers in the 0-1 range, then negatives. When this is not appropriate, we should make clear the limitations of our examples.

Often instructions about what **to do** are made explicit in a classroom. For example, a teacher models how to factorise a quadratic as a step in solving a quadratic. However, **decision-making** and **mathematical reasoning** are often less explicit; for example, teachers may not articulate/draw attention to what prompts the attempt to factorise as opposed to another approach to solving. (When planning, you should script key explanations – ensure you **script** the why (decision making) as well as the what (doing) – thinking aloud approaches).

The situation is then complicated because there are many different aspects of the problem that can be attended to**.** In the quadratics example this could include: recognition of the equation as a quadratic, as opposed to linear, cubic etc.; understanding what 'solve' is asking you to do; recognising assumptions and implications, such as $\mathbb{R}$ or $\mathbb{Z}$ for how many solutions; running through the different methods; thinking about layout; affordances of different representations; comparison of methods and their efficiency in given situations...

Notice-Expert – Attention and Flexible Thinking

In addition to noticing differently, an expert is likely to have intuitive shifts in attention (Stylianou and Silver, 2004) with 'automatic access' to a range of knowledge, skills and understanding that allows for flexible thinking, leading to appropriate solution strategies.

For example, in the quadratics example, an expert may have a mental 'play' with the coefficients to consider if it is factorisable or not, whilst also keeping in mind other potential strategies and their respective ease/hardness. All of this might be completed intuitively and in a short space of time (e.g. when reading the question). In the fractions example, an expert may shift between the part-whole model with division (2÷3) and magnitude (a bit more than half) depending on what is being asked. For an expert, all of this shifting attention may occur so automatically that it goes unnoticed and therefore often unshared.

We, as teachers/experts, have good intuition as to what to attend to mathematically – we need to think carefully as to how to make this accessible to others. Whilst 'telling' can be part of this process, it cannot be all – recognising where our own attention lies is difficult, recognising it in others even more so. Supporting learners to develop their own mathematical recognition has to be linked to their doing of mathematics; this is hard when students are grappling with unfamiliar concepts and with many potential facets to attend to.

2.3.3 The Power of Theories – Enhancing Your Knowing

The mind, even with developments in neuroscience, is not yet fully understood. Therefore, we have models to provide insights into how the mind might work, such as modelling memory with short and long-term elements. Learning, as a complex 'hidden' construct, needs models or theoretical frameworks to interpret observable behaviours. However, no one single theory or model is sufficient to explain all about the mind; no matter how well-founded, theories are open to challenge and/or development.

There are many 'theories of learning', which seek to articulate how we learn. One 'simple' challenge is that two people presented with the same stimuli learn different things. There are debates about the relationships between learning, memory, (prior) knowledge and understanding. Some research is conducted in laboratory settings, with features explored in isolation –understanding implications for classroom contexts is far from straightforward due to the multiple interrelated 'variables' and social interactions (EEF, 2021). Some theories have offered significant insights into the mathematics classroom, but even these do not capture all perspectives and do not encompass all contexts and situations.

Then there are 'theories of pedagogy'; i.e. how we generate good learning opportunities through how we teach - linked to, but not the same as, theories of learning. And, as above, learning to teaching is a complex undertaking with many facets.

When you try to understand classrooms, thinking and learning cannot be directly observed. Consequently, you need to make inferences (interpretations) based on behaviours that you notice + model(s) that you hold, implicitly or explicitly. For example:

> A) (Behaviour): You observe a student staring out a window.
> (Interpretation): Are they (i) thinking hard about what you have just said or (ii) is their attention elsewhere or (iii) a reason not thought of?

You may draw on your knowledge of the student, the cultural, classroom norms, … You may make this judgment and respond at an intuitive level (almost without noticing), such as tapping their desk whilst you continue. Or something might prompt you to take more deliberate action (Watson, 2019b).

> B) (Behaviour): You demonstrate a method for solving an equation. You notice a student has produced a correct solution to the first question. I.e. performance: demonstrating a particular skill
> (Interpretation): Did they (i) produce the solution with 'local' knowledge (following the routine from the modelled example), (ii) did they recall from prior experience, (iii) did they understand the concept in a way they could apply in range of contexts without the support of the modelled example, (iv) or…

We need to understand the models/assumptions/beliefs we hold that are used to make those interpretations.

Student Reasoning vs Mathematical Horizon.

When students respond, rather than exploring and seeking to understand student reasoning, teachers have a tendency to focus on 'their mathematical horizon' (Ball, 1993; Baldry, 2019), namely the methods/ concepts they would like students to adopt. Whilst the latter is an appropriate longer-term goal, it is argued that a more balanced approach where student reasoning is explored more of the time would be of benefit.

The notion is that student reasoning/responses will be based on something that is rational to student, as they will have built up mental models (their schema). It is a rational decision, based on their prior knowledge and understanding, though this could be related to social cues rather than mathematics. (A student's social standing is likely to be more important to them than the mathematics, so 'least embarrassing' might be the driver). In order to understand this reasoning, for both 'correct' responses and 'errors', we need to provide the space for students to express their ideas. This is even more important with potential misconceptions – in order to provide opportunities to restructure erroneous mental models we need to understand the structure that is currently held first.

The 'mathematical horizon', the logic of the discipline, your reasoning: Directly explanation plays its part in student learning, but it is less likely to challenge mental models held. The students need to do mathematics; to become self-regulating learners they need ways to check their thinking against the discipline (rather than rely solely on teacher evaluation).

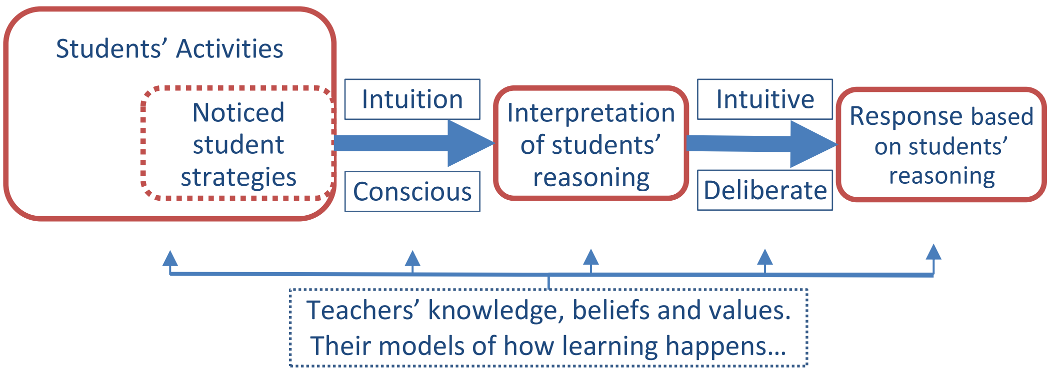
Greater levels of exploring student reasoning should also aid assessment for learning (A4L) and further the development of student reasoning. It provides opportunities for students to articulate their reasoning, developing appropriate vocabulary, whilst also allowing the teacher to draw attention to mathematical decision making and mathematically salient features.

Professional Noticing

In order to have an impact on students, noticing, interpretation and response has to happen 'in-the-moment' of lessons, when there are many things happening at one time. Jacobs et al. (2011) and van Es & Sherin, (2008) articulated a three-stage process related to students mathematical thinking:

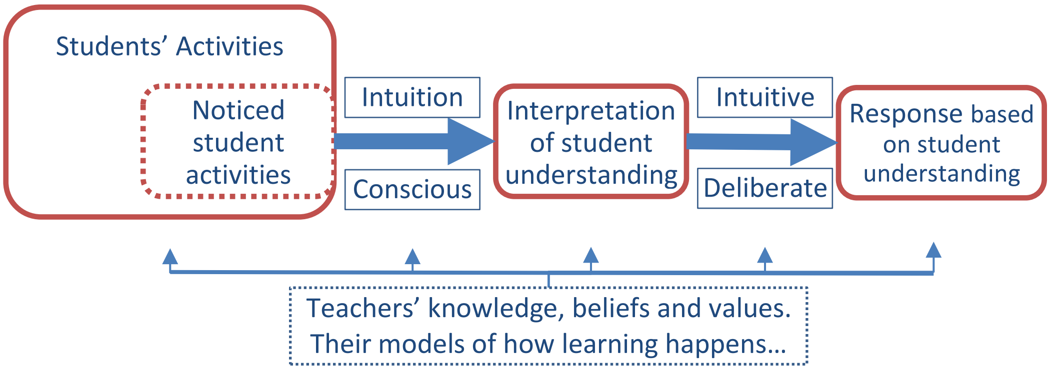
- Attending to students' strategies
- Interpreting students' reasoning/understandings
- Deciding how to respond on the basis of students' reasoning/understandings.

Assessment of Student Reasoning



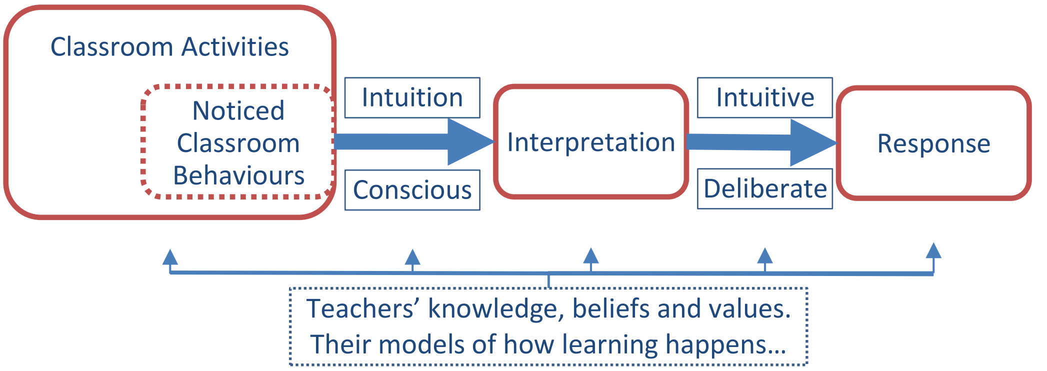
Interpretation of classroom activities

Student Understanding



Classroom Activities

This model can be extended to a wider set of behaviours. At each stage, cognitive responses may be intuitive or more consciously deliberate:



Multiple time during lessons a teacher will notice classroom behaviours, interpret those activities and respond. At each stage, a teacher's knowledge, beliefs and values influences what is noticed, how those activities are interpreted and what action is then taken. As outlined above, many elements of decisions may be made below a 'consciously deliberate' level – in these circumstances, it is even more important that teachers have a well-founded understanding of learning, as intuitive actions are highly influenced by beliefs and values.

Affect, motivation and other 'hidden' constructs

Furthermore, there are many other aspects of classrooms that are only available indirectly, such as students' motivation, the impact of societal and classroom norms or teacher's decision making.

Inferences

So, theories and models should be well-founded and useful, providing insights into complex situations that are difficult or impossible to access directly. This allow better founded inferences to be made, with these interpretations leading to better founded responses. As learning is a complex 'hidden' construct, some reference to theoretical/conceptual frameworks (implicitly or explicitly) is required to in order to make these inferences. However, no one theory explains all… , so maintain a critical stance.

2.3.4 Classroom Implementation

Linking Tasks to Learning: Articulating Mathematics Concepts

A key question for the teaching of mathematics is the relationship between demonstrating a **skill** (doing, performance) and **learning**:

> What is the relationship between ***tasks*** set (what the students are *doing*) and mathematical **learning** opportunities inherent in the task?
>
> In order to have some ideas of what students *might **learn*** about mathematical concepts, structures, meaning and/or ways of working by engaging in the task, you first need to be able to articulate the mathematical **concepts** involved – this is harder than it looks.

When we interpret classroom activities, not only do we need to consider the potential relationship between performance (demonstrating a skill - doing) and understanding, we need to consider the level of understanding. For example, would solution strategy used in a lesson following the modelling of a similar example be remembered and recognised later? In other words, what is the likelihood that that skill will transfer beyond that particular activity?

Understanding potential links between **doing**, **learning, knowing** (inc. memory, and recall) and **understanding** is far from straightforward, but this is what makes teaching so interesting.

One categorisation you might find useful is **knowledge, skills** (doing) **and understanding.**

For example, solving a linear equation might be done by following a procedure modelled by the teacher.

> The student does not need to be able to identify the type of problem or complete independently.

To 'understand' involves many mathematical concepts:

> equivalence, the role of =;
> recognising (separating) linear equations from expressions and equations of other orders;
> the role of letters, including what 'solve' means;
> associative, commutative and distributive properties;
> works for any real number;
> …

One version of 'know' involves recognition, remembering and recall at the appropriate time. Another version of 'know' is to recall isolated facts when given particular prompts. So, as with learning, the definition and meaning of 'know' needs to be considered.

Where is the mathematics?

Always ask: "Where is the mathematics"

Articulate the **link** between **task(s)** and **mathematical learning***

i.e. how might *doing A* lead to *learning B.*

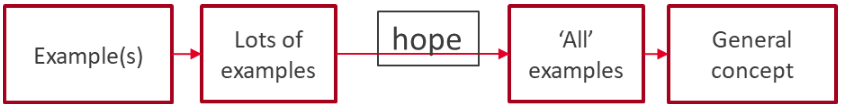
What *mathematical concept(s)/reasoning may be met by engaging in the task and how: *Identifying the mathematical intention of the task and how is it to be enacted* requires you to articulate your pedagogical reasoning and your models of the learning of mathematics as well as the mathematical concepts involved.

A common approach is:

(We would argue this often leads to missed opportunities)

This approach often consists of the teacher demonstrating an example (describing the 'doing', without necessarily modelling mathematical decision-making or reasoning). After which the students practise other similar examples.

This can lead to mathematical decision making residing with the teacher, with students completing tasks by 'rote' without necessarily making links to mathematical principles. In other words, it is more in hope than design that students make links between the examples met and general concepts.



A few 'tweaks' to this approach (see below) can transform students' access to mathematical concepts and the logic of the discipline, with the key often the 'interrogation' phase.

'Better' approaches: Include Interrogation

Modelling: "I do, we do, (you do)"
Model reasoning and decision making *(thinking aloud)* as well as process [when planning **script**].

Consider partial modelling

e.g. gaps, just the diagram,...

Encourage student explanations

e.g. write in silence and ask for a description, ask why, what next, link diagram ⇔ algebra/sum...

Consider modelling two: the start of variation/comparison.

This can be effectively combined with student explanations (thinking aloud) – what is the same/different, when would (a) be easier/harder than (b),...

Practice: "You do"
Careful consideration of the examples/tasks that students meet and their **sequencing** (variation) can help draw students' attention to particular features during completion (Watson, 2017), and also provide opportunities in the interrogation phases to make connections.

Here, the choice of examples (**details** matter, such as what numbers chosen), representation(s) and sequencing matters; **what** is varied and **how** matters.

Interrogation: Use (value) the work done by pupils before moving on.
The aim is to expose mathematical structure and to draw attention to mathematical critical features and concepts. In order to do this, it is vital that you have articulated the mathematical concepts involved in the task.

Example of Interrogation include:

- Expand students' example space (the range of examples a learner associates with a concept) by asking:
  - *What would happen if* I changed… ;
  - What would I have to change to make this… ;
  - Would this work for… ; does the approach always work, does it every 'break down'
  - Why is this one easy and this one hard;
  - Is the solution unique (how do you know); …
- Student generation:
  - Write you own questions that is:
    - Similar to; with an answer of; uses…
    - Hard, easy; a good illustration of…; .
  - Self-explanation:
    - Students engage in different forms of self-explanation e.g. write an explanation to introduce this topic to a group in the year below…
- Compare and/or make links to alternatives
  - Evaluate effeteness and scope of alternative approaches
  - Used different representations, concepts and contexts.

See Prestage and Perks (2013) for further ways to enrich 'standard' activates with 'tweaks'.

Models e.g. Silent e.g. Two Parallel → Student Explanations → Students Practise → Interrogation

Interrogation can happen at a variety of stages and levels of interaction. Such as one-to-one conversations as you circulate the room, as part of mini-plenaries part way through a task, before students continue, as you introduce 'the next step', plenaries or reviews at the start of a subsequent lessons.

Recognising the Type of Problem (starting with 'X' and 'not X')

However, students also need to be able to **recognise** the nature of the problem for themselves (rather than being told by the teacher, the title or the heading in the textbook)

What is this? → Mixture of 'X' and 'not X' → Student Decision Making → Approaches and representations

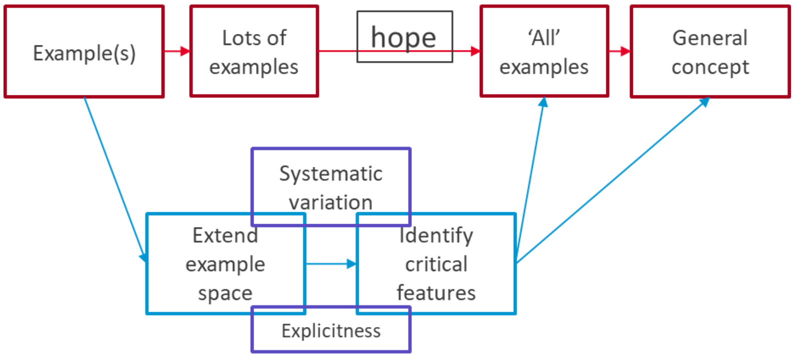
One common approach is to mix in the type of examples with other 'not examples' (e.g. quadratics with linear & cubic).
A good mix of examples might include: typical, atypical, boundary and 'not' examples (but with similar surface features)

e.g. $x^2 - 4x + 6 = 0 \quad 4x^2 = 9 \quad x^2 - 4xy + 6 = 0$

This is another way to extend a student's understanding of an example space for a particular concept.

Examples

Examples form an important role in teaching mathematics, and it is important to think about how to support students to see the **general through the particular.** In particular, offering ways to recognise the mathematically significant features of examples. Here the **sequencing** of examples and the **range** of examples met is key (see variation theory for further details).



Some of this can be achieved through 'tweaking' what you ask and when, rather than major changes. I.e. How you use the work that the students have done to draw attention to key features during the interrogation phases can have a significant impact on how pupils learn from examples.

There are other approaches: Problems

Problem → Mathematical tools → Application to a wider field

Here a problem is presented that requires the students to develop their understanding of new (to them) mathematical tools. That said, finding such problems is not simple. For example, Pythagoras could be introduced this way if you found a scenario where scale drawing was insufficient.

And then there are 'rich tasks' and 'rich examples':

Rich task → Draw attention to key featurees

Rich tasks/examples often have feedback 'built in' and/or are structured so that the students meet the mathematical ideas as they complete the activity (rather than being shown beforehand which they then practice).

For example, the multiplicative rather than additive relationships in enlargement being explored through whether pictures 'deform' (circles work well), or ICCAMS lessons (see Blackboard) with ratio explored through elastic bands. Technology can also be used to integrate feedback into the task (e.g. dynamic geometry).

Problems that have '**low floor** and **high ceiling'** could fall in this category. Meaning all students should be able to start and undertake legitimate mathematical activities, but the task can 'expand' to provide alternative challenges. For example, the A4 rotating paper octagon problem – one approach can be measuring angles, another involves ratio, algebra and Pythagoras, and notions of proof can also be drawn in.

However, finding these rich tasks is not easy. And you need to ensure that you have fully investigated these activities, because a critical role of the teacher in these situations is to draw attention to key mathematical features that might otherwise go unnoticed by students.

2.3.5 Key Ideas and Theories

Mathematics educational research is a huge field. The following represents some key perspectives that will be explored in the course– it is intended as a starting point not an exhaustive list:

Variation Theory and the Role of Examples;

APOS Abstraction (concrete pictorial abstract)

Multiple Representations (Berthold, 2009)

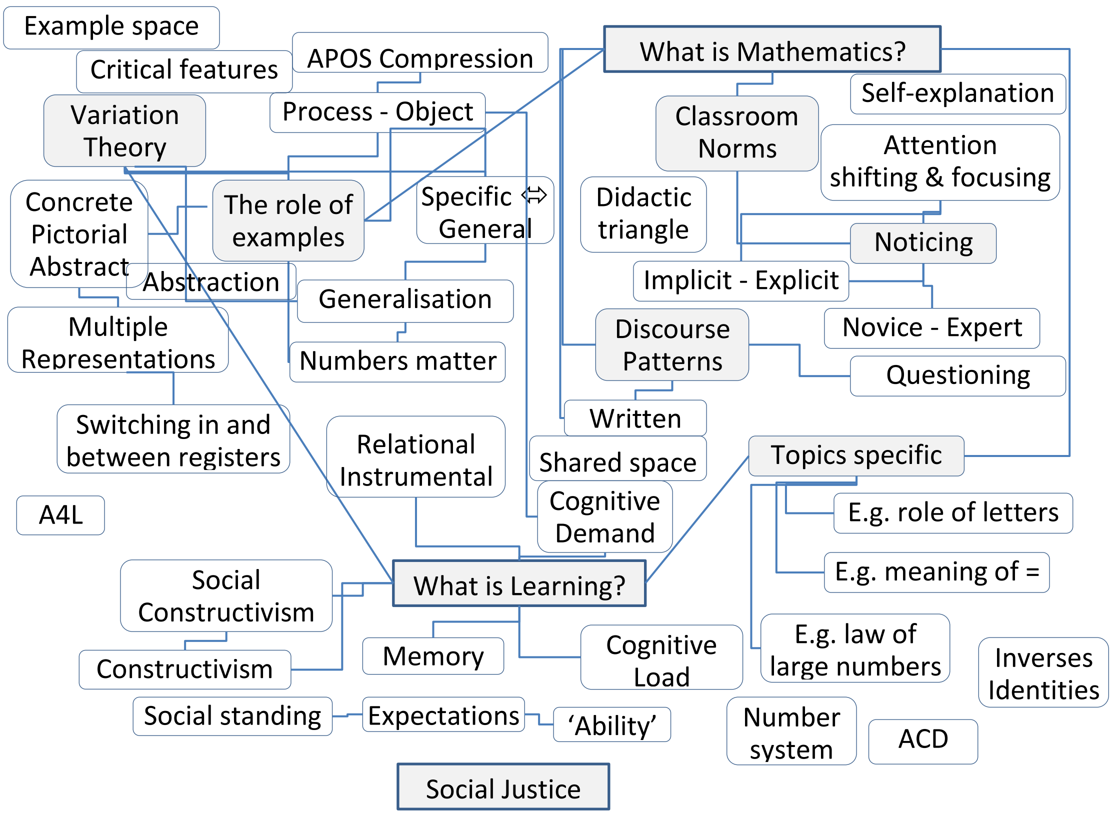
Cognitive Load

Topic Specific (e.g. = sign)

Classroom Norms and Interactions – attention and explicitness.

Constructivism/ Social Constructivism

As the course progresses, you should be developing your own 'map' as to how different theories and perspectives offer insights into your interpretation of sources (e.g. CCF and research) and your practice. The following map is not intended for you to directly access; instead, it is an extract of my curriculum plan indicating the complexity of the field.



The next section (2.4 Theoretical Perspectives) will link to material on Blackboard and may expand as the course progresses.

2.4 Theoretical Perspectives

As the course progresses, you will access materials on Blackboard (e.g. PowerPoints) and some summaries of key ideas may be added here.

Neuroscience
- Cognitive Load
- The Cognitive Load and Direct Instruction Debate: (Sweller et al., 1998; Kirschner et al., 2006; Hmelo-Silver et al., 2007; Sweller et al., 2019; Watson, 2019a; Sweller, 2021)
- Building cohesive mental models

Pedagogy
- Rosenshine, Mastery, Direct Instruction
- Scaffolding (Anghileri, 2006; Bakker et al., 2015)

Mathematics Theories of Learning
- Variation Theory and the Role of Examples;
- Process-Object
- APOS Abstraction
- Concrete pictorial abstract
- Multiple Representations (Berthold 2009)

Mathematics
- Arbitrary Necessary
- Generalisation ("Without generalisation there is no mathematics" Mason (2018)
- Proof

Topic Specific
- Statistics
- Probability
- Number (e.g. = sign)
- Algebra

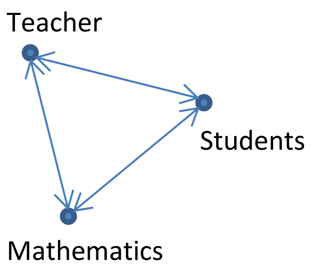
Constructivism/ Social Constructivism
- Noticing – Attention, Professional noticing
- Classroom Norms and Interactions – attention and explicitness.
- Discourse
  - Questioning and funnelling
    - The dilemma of telling Mathematical horizon – student reasoning

Numeracy and decision making: (Peters, 2008).

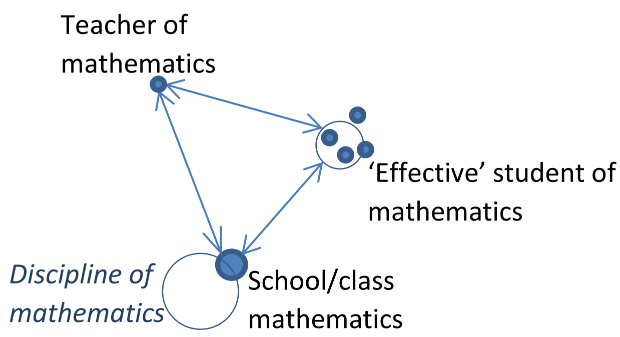
2.5 Published Frameworks

2.5.1 Didactic Triangle: Teacher-Students-Mathematics

Classrooms are complex, dynamic environments, and the multifaceted nature of classroom interactions means that there are no simple ways to model the teaching and learning of mathematics, but one common starting point is the ‘didactic triangle’.



(Baldry 2019, adapted from Straesser, 2007, p.165)



Focussing on different elements of triangle can help consider the classroom from different perspectives – prompting questions such as ‘how does the student experience …’. It may also help us understand how different theories might ‘fit together’. For example, teachers’ subject knowledge would focus on teacher-mathematics, and theories of learning would be students-mathematics, whilst pedagogy is how the teacher orchestrates the mathematics for the students.

This section looks at the classroom from different perspectives, drawing on some influential frameworks. However, it should be remembered that these are interconnected and cannot be treated in isolation and they only represent a small fraction of the many other theoretical perspectives ‘out there’.

2.5.2 Mathematics-Students: The National Curriculum

The National Curriculum is more than suggested content; it provides a perspective on the nature and the learning of mathematics – as such, it offers a theoretical framework for a mathematics curriculum. Whilst no scheme is perfect, it is referenced here as it provides a coherent structure that moves beyond particular skills or content. The following are extracts from Key Stage 3 – please see the full document for the content element and KS4.

Purpose of study

Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history’s most intriguing problems. It is essential to everyday life, critical to science, technology and engineering, and necessary for financial literacy and most forms of employment. A high-quality mathematics education therefore provides a foundation for understanding the world, the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject.

Aims

The national curriculum for mathematics aims to ensure that all students:

♣ become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that students develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately.

♣ **reason mathematically** by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language

♣ can **solve problems** by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication, including breaking down problems into a series of simpler steps and persevering in seeking solutions.

Mathematics is an interconnected subject in which students need to be able to move fluently between representations of mathematical ideas. The programme of study for key stage 4 is organised into apparently distinct domains, but students should develop and consolidate connections across mathematical ideas. They should build on learning from key stage 3 to further develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems. They should also apply their mathematical knowledge wherever relevant in other subjects and in financial contexts.

The expectation is that the majority of students will move through the programme of study at broadly the same pace. However, decisions about when to progress should always be based on the security of students' understanding and their readiness to progress. Students who grasp concepts rapidly should be challenged through being offered rich and sophisticated problems before any acceleration through new content. Those who are not sufficiently fluent with earlier material should consolidate their understanding, including through additional practice, before moving on.

Working mathematically (Key Stage 3)

Through the mathematics content, students should be taught to:

Develop fluency

♣ consolidate their numerical and mathematical capability from key stage 2 and extend their understanding of the number system and place value to include decimals, fractions, powers and roots

♣ select and use appropriate calculation strategies to solve increasingly complex problems

♣ use algebra to generalise the structure of arithmetic, including to formulate mathematical relationships

♣ substitute values in expressions, rearrange and simplify expressions, and solve equations

♣ move freely between different numerical, algebraic, graphical and diagrammatic representations [for example, equivalent fractions, fractions and decimals, and equations and graphs]

♣ develop algebraic and graphical fluency, including understanding linear and simple quadratic functions

♣ use language and properties precisely to analyse numbers, algebraic expressions, 2-D and 3-D shapes, probability and statistics.

Reason mathematically

♣ extend their understanding of the number system; make connections between number relationships, and their algebraic and graphical representations

♣ extend and formalise their knowledge of ratio and proportion in working with measures and geometry, and in formulating proportional relations algebraically

♣ identify variables and express relations between variables algebraically and graphically

♣ make and test conjectures about patterns and relationships; look for proofs or counterexamples

♣ begin to reason deductively in geometry, number and algebra, including using geometrical constructions

♣ interpret when the structure of a numerical problem requires additive, multiplicative or proportional reasoning

♣ explore what can and cannot be inferred in statistical and probabilistic settings, and begin to express their arguments formally.

Solve problems

♣ develop their mathematical knowledge, in part through solving problems and evaluating the outcomes, including multi-step problems

♣ develop their use of formal mathematical knowledge to interpret and solve problems, including in financial mathematics

♣ begin to model situations mathematically and express the results using a range of formal mathematical representations

♣ select appropriate concepts, methods and techniques to apply to unfamiliar and non-routine problems.

Mathematics guidance: Key Stage 3

Non-statutory guidance for the national curriculum in England (September 2021)

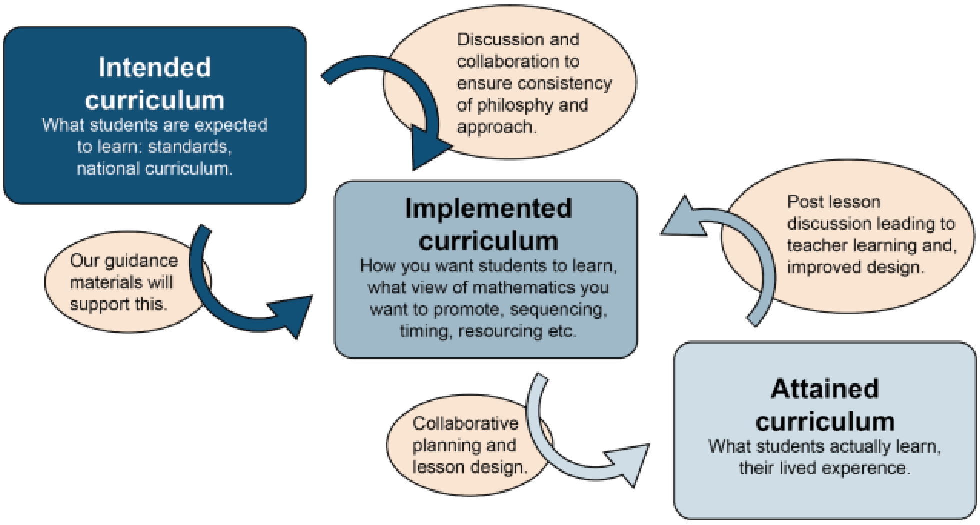
https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/1056795/KS3_NonStatutory_Guidance_Sept_2021_FINAL_NCETM.pdf

This government document provides detailed guidance that dovetails with the guidance for primary schools. You should become familiar with the document and approaches described. Make sure that you follow a topic through in detail in order to develop an understanding of the potential of these ideas to support mathematical thinking. In particular, note the language and representations that are recommended.

The authors also highlight a useful approach about how to think about curriculum (DfE, 2021, p.14) – this guide uses the terms intention, implementation and impact:

It is helpful to consider three different embodiments of a curriculum, the:

- intended curriculum
- implemented curriculum (the school curriculum or scheme of work)
- attained curriculum.



2.5.3 Students: What is an Effective Student of Mathematics?

As discussed earlier, we as teachers communicate to students what is considered to be a 'good student' through our actions. For example, if we praise short, correct answers then an effective student is likely to be perceived as one who can provide those answers accurately and quickly. Consequently, it is worth articulating what you consider an effective student of mathematics should 'look like' in your classroom and shape classroom activities, interactions and your responses around that vision.

Kilpatrick et al. (2001, p.116) outlined 5 strands they argued a mathematically proficient learner would need – I find the productive disposition a very useful notion:

> Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully. Mathematical proficiency, as we see it, has five components, or strands:

> Conceptual understanding—comprehension of mathematical concepts, operations, and relations
>
> Procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
>
> Strategic competence—ability to formulate, represent, and solve mathematical problems
>
> Adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
>
> Productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

These strands are not independent; they represent different aspects of a complex whole… the five strands are interwoven and interdependent in the development of proficiency in mathematics.

Relational and Instrumental Understanding

The notion of relational versus instrumental understanding introduced by Skemp (1976) is an often cited. (Relational: knowing both what to do and why. Instrumental: rules without reasons). The former is often stated as a teacher's aim, but a close interrogation of student experience often raises a question as to whether the latter would more closely match the classroom requirements of an effective student of mathematics, as communicated through classroom interactions and norms.

Engagement

Student engagement is often talked about, but engagement has lots of forms and, as with learning, is not easily discernible. One common starting point is to think about engagement from three perspectives:

cognitive, emotional (affect) or behavioural (operative).

"Student engagement in learning experiences is understood to relate to…:
cognition (think hard),
affect (feel good),
operation (work toward being more productive learners)
Engagements that need to be occurring simultaneously at high levels" (adapted from Xu and Zammit, 2020, p.3).

An effective student of mathematics in my classroom will be…

Two key questions are: (i) What is your vision of an effective student of mathematics (ii) What actions are you going to take to communicate that vision to your students. Look back over section 2.5.3 and draw together your key ideas.

2.5.4 Teacher: Knowledge

Pedagogical Content Knowledge

The role of subject knowledge for teaching is much discussed. There are two common threads to these debates:

(i) Teacher knowledge has a substantial impact on teaching

(ii) Knowledge extends beyond personal knowledge of the subject.

The categorisation of teacher content knowledge introduced by Shulman (1986, p.10) has been widely used and developed:

Subject Matter Knowledge: Content Knowledge - amount and organization of knowledge *per se*.

Pedagogical Content Knowledge: Subject knowledge for teaching. This includes knowing representations and formulations that make the subject comprehensible to others (e.g. the most useful representations, … examples, explanations), and what makes specific topics easy or difficult to learn.

Curricular Knowledge: Familiarisation with curriculum materials and examinations.

Ball et al. (2008, p.5) added further subcategories:

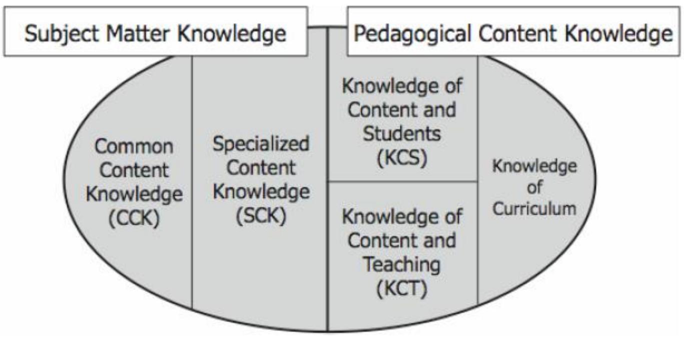


Figure 2. Shulman's Original Category Scheme (1985) Compared to Ours

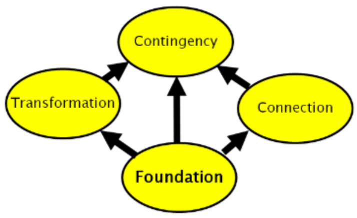
These categories inevitably overlap but they highlight the interrelationship between the teacher, the students and the mathematics.

Knowledge Quartet

Work by Rowland (and colleagues) on the **Knowledge Quartet** (KQ) offers a framework as to how "knowledge can be seen in the act of teaching" (Rowland et al., 2005; Rowland et al., 2015). Whilst this work originated with primary teachers, the research has extended into secondary school settings.

Foundation: Everything brought to the planning of lessons – knowledge, beliefs and values.

Transformation: Transformation of knowledge into actions (e.g. what examples, what representations…), as demonstrated in planning and in the act of teaching.



> PCK: "…the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful" (Shulman, 1987, p. 15).

> Connection: Coherence within and across lessons – the sequencing of topics drawing on knowledge of structural connections and cognitive demands.
>
> Contingency: Responsiveness - the readiness to respond to students' ideas in the lesson.
>
> (Adapted from Knowledge Quartet, 2020)

Knowing – Doing – Being

One perspective is that Shulman (1986) and Ball et al. (2008) are articulating a model for teacher 'knowing', whereas the work by Rowland et al., (2015) encapsulates 'knowing' in their foundation category but include 'doing' in the remaining elements.

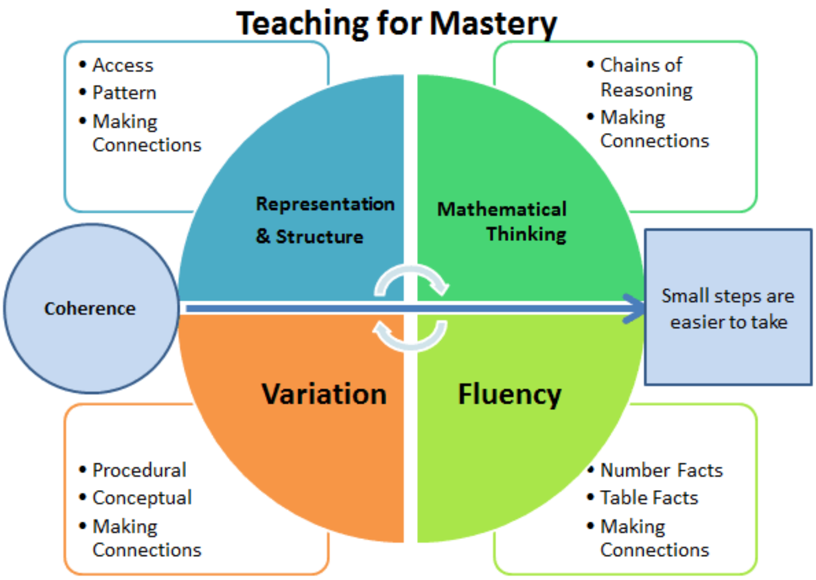
As articulated in section 2.1, we adopt a knowing, doing, being approach to our curriculum, with the interrelated and reflexive nature of these components recognised.

2.5.5 Teacher & Mathematics ⇔ Students: NCETM and Mastery

Care is needed with the term 'mastery', as it is used by different people/organisations in different ways. E.g. Jerrim et al. (2015, p.5) argue mastery programmes aim to:

> Deepen students' conceptual understanding of key mathematical concepts. Compared to traditional curricula, fewer topics are covered in more depth, and greater emphasis is placed on problem solving and on encouraging mathematical thinking

NCETM's approach is important as this is a key element of the Maths Hub programme (government funded CPD 'what works' network). Many primary schools have been involved in mastery training and there are also programmes focussing on KS3/4, FE.



(NCETM, 2017, p.1) https://www.ncetm.org.uk/

The explanations listed on the website by NCETM (2017, p.1) are:

> Coherence: Lessons are broken down into small connected steps that gradually unfold the concept, providing access for all children and leading to a generalisation of the concept and the ability to apply the concept to a range of contexts.
>
> Representation and Structure: Representations used in lessons expose the mathematical structure being taught, the aim being that students can do the maths without recourse to the representation
>
> Mathematical Thinking: If taught ideas are to be understood deeply, they must not merely be passively received but must be worked on by the student: thought about, reasoned with and discussed with others
>
> Fluency: Quick and efficient recall of facts and procedures and the flexibility to move between different contexts and representations of mathematics
>
> Variation: Variation is twofold. It is firstly about how the teacher represents the concept being taught, often in more than one way, to draw attention to critical aspects, and to develop deep and holistic understanding. It is also about the sequencing of the episodes, activities and exercises used within a lesson and follow up practice, paying attention to what is kept the same and what changes, to connect the mathematics and draw attention to mathematical relationships and structure.

Variation Theory is a key theory of learning for mathematics will be explored in more detail in the course. As with the term ‘mastery’, different people/organisations use this term in different ways.

2.6 Groups: 'Categories' of Students

The Teachers' Standards refer to 'the needs of all students, including those with special educational needs; those of high ability; those with English as and additional language; those with disabilities…' (DfE, 2012, p.1). Also, schools are granted additional funding for students entitled to Pupil Premium (criteria based on proxies for low socioeconomic backgrounds). Both teachers and schools are expected to account for the progress of students seen as members of these groups. The breakdown of results by other categories, such as gender and ethnicity, are also reported, and educational research includes work focussed on the learning and educational progress of different groups of students.

However, when referring to groups of students we need to take care as to how we use language. This is particularly important in situations where we are talking about individuals.

For example, the statement 'boys like competition' is problematic on number of areas. First, this implies it applies to *all* boys (and possibly *no* girls). Second, you would need very strong evidence to back up such a general statement – I would argue this evidence does not exist. Even if there was a measure, such as a survey, that indicated more boys liked competitive games as compared to girls, this cohort characteristic should not be applied to an individual student, be they a boy or a girl, as individual differences would outweigh the cohort measure. Moreover, before this was applied to a classroom context, links between 'like competition' and learning would have to be established (and indeed, there is evidence that competition detracts from learning goals…).

- In general, there may be measurable differences between cohorts, such as 'girls prefer non-STEM subjects', but we should not apply cohort characteristics to individual students.
  - For the same reason, we should not apply progress/attainment 8 cohort information to track/ 'target' individual students – even the DfE agrees with this.
- Think about the difference between: "a SEND student" and "a student with a SEND"
  - The latter is far more appropriate
- When applying to cohort measures to classroom contexts, we need to ensure that links to learning are established (which is, again, harder than it looks).

In a few assignments in the past, some student-teachers have made or implied **inappropriate** associations between particular groups. For example, some have discussed students with SEND or EAL in the same phrase as 'low ability', or have associated behavioural characteristics with attainment groups.

There is robust national and international research evidence that, from a cohort perspective, students from low socioeconomic backgrounds 'do less well' in educational terms. Schools will have policies related to Pupil Premium and the extra funding this entitlement brings. Student related explanatory factors, such as less academic support from their home environment, have been proposed in research, but we need to take care as school and societal factors have also been identified (e.g. organisational issues such as school allocation, curriculum access, class organisation and staffing). One key issue is that evidence indicates that teachers' expectations and (unconscious) bias impacts on learning opportunities for students and attainment outcomes (e.g. de Boer et al., 2010). So we need to take care labels, such as Pupil Premium, do not lead to lower expectations for some students. We can think that we are being helpful, when in fact our actions have the opposite effect (e.g. praise for work a student considered easy or did will little effort can lower a student's academic self-concept).

> In general, we need to be as conscious as we can about assumptions and stereotypes that we might hold, and the effect that our expectations can have on the learning opportunities we make available to different students. *Engaging with research literature is a key amelioration strategy.*

2.6.1 'Ability' and Setting

Notions of 'ability' are particularly relevant in mathematics because of the prevalence of setting. Whilst the language of 'setting by ability' is common in schools, this use of language is challenged in educational research from a number of perspectives. For example, Wiliam and Bartholomew (2004) argued 'we believe that such notions of ability are not in any way well founded and are of dubious validity as predictors of potential' (p.281). Also 'ability' can be associated with low expectations and notions of 'fixed potential', and limits are often imposed some students' access to the curriculum (e.g. foundation syllabus). Schools that set often draw on measures of *prior attainment* (tests/exams), so it is preferable to use this term rather than 'ability' in academic settings (including mathematics PGCE sessions and assignments).

Whilst setting is still common in English schools, research indicates that this does not improve overall attainment and actually embeds inequity, as allocation to low sets tends to depress those students' attainment. Historical curriculum developments, including the stratified exam curriculum (foundation and higher tiers), means that grouping students by measures related to attainment has long been a tradition in England, but over recent years there has been some pressure to shift away from a 'setting' approach. The focus on 'mastery' in primary schools has potentially reversed some of the grouping of students in that sector, and there is some anecdotal evidence that some secondary schools are (re)introducing mixed attainment classes in the lower years and/or banding students into broader groups. However, significant curriculum and professional development, alongside changing parental expectations, would be needed for widespread change.

With stratified schemes of work, students placed in lower attaining sets often experience a more limited curriculum, thus limiting learning opportunities and making any set moves more problematic. Research indicates setting is rarely based purely on measures of attainment, with teachers/schools 'overriding' attainment scores based on other (potentially unconsciously biased) judgments, which disproportionally impacts on already more vulnerable groups (Wiliam and Bartholomew, 2010). In particular, research indicates students from low socioeconomic backgrounds tend to be over represented in low sets, even when prior attainment is taken into account (e.g. Francis et al., 2019). Issues of inequity are compounded as movement between sets tends to be infrequent; few escape sets with low attainment profiles once placed there.

Expectations and language can impact in multiple ways; why are we surprised at apparently lower engagement from students place in lower attaining sets when those students are often presumed to be less capable of working independently and cooperatively, and phrases such as 'bottom set' and 'low ability' are commonplace (e.g. Boaler, 2005).

Please ensure you pay close attention to your language, and consider carefully any assumptions you carry.

3. Curriculum Planning and Review

Please ensure you have completed the Algebra Equivalence task (see section 4)

3.1 Plan and Review Each Lesson

You should **plan** and **review** (evaluate) **each lesson** that you teach, but how you plan and review will evolve over time, as described below.

Golden Rule: You **must** complete **all tasks** that you are going to give to the students.

Planning and preparation for teaching are essential. They do not guarantee perfect lessons but they help you to avoid disasters (or at least help you identify quickly that a lesson is going awry and provide an escape route). Good preparation helps you feel more confident, particularly when you are under pressure in the classroom, and allows you to adapt plans and respond flexibly as you assess student activity.

Schools should have a policy regarding planning and reducing teachers' workload, which could involve joint/shared/collaborative planning. Experienced teacher may be able to plan for lessons without writing individual lesson plans *but you will need more time to think and you will write more* regardless of the planning approach you are taking.

3.1.1 Collaborative planning and review:

Collaborative planning can be a very effective way to develop your understanding, as you need to articulate your thinking and consider others' points of view. However, it does take time, especially when first undertaken. One effective way to be inducted into lesson planning in your particular school's context is for your co-tutor/mentor to model their lesson planning. For example, by verbalising their thought processes as they plan a lesson, after which you observe that lesson. **If possible:** undertake some collaborative planning with your co-tutor/mentor, class teachers or peers, and where possible include lesson observations and joint review – part of the CCF 'Observing how expert colleagues … and deconstructing this approach' (but remember teachers may struggle to find the time).

3.2 Overview of the Planning Processes

3.2.1 Long-, Medium- and Short-Term Planning

Planning is often considered at different scales: short, medium or long term.

<u>Long-term</u> plans are usually departmental schemes of work, broken down into years. These usually show which units of work will be covered and when.

<u>Medium-term</u> plans (topic/unit plan) outline the teaching over a series of lessons, usually related to a particular topic. This planning should reflect key concepts and their development, the prerequisites for learning and links to future concept development, and ideas for resources and approaches to teaching.

<u>Short-term</u> plans are often considered to be a lesson plan, though this may extend over a small number of lessons. This should demonstrate how you are going to adapt a medium-term plan for your particular class.

As you are new to the profession, your plans should include more detail than an experienced teachers would write.

3.2.2 Format: Medium to Short term

Initially you will need to write individual lessons plans; a lesson plan pro forma is available on Blackboard (also, see section 5), which includes examples and prompts. You or your placement school might wish to use a different format; you can do this *as long as this contains all the information that is on the UoL pro forma* and your school co-tutor/mentor agrees.

You can hand write or produce electronic versions – whichever is most efficient for you. (Lesson plans written in pencil are very flexible when used in conjunction with an eraser! – scanning/photographing and uploading plans of observed lessons to Eportfolio is quick)

As you progress through the course, and with the agreement of your co-tutor (mentor)/UoL tutor, you should be aiming to streamline the planning and review process. However, this means reducing the amount of time spent writing about routines, *not reducing the thinking about or the doing of the mathematics.* (See 3.5.5 regarding PowerPoints)

Planning is most effective (both in the use of your time and in terms of student learning) if **medium term** planning is done first, followed by individual lesson planning. However, when you first start you are likely to plan lesson by lesson. **Initially**, many find it harder to medium term plan first, but the sooner you can switch to this approach the more time you will save. So, over time, you should be moving to medium term planning as your main planning approach (in conjunction with an abbreviated form of individual lesson plans).

In addition, elements of your plans may appear in different locations. For example, a well annotated worksheet (identifying multiple solution strategies, common misconceptions, barriers, key questions…) may replace some of the pro forma. Details about adaptions for SEND may be in your medium-term plan and/or teaching file. However, moves away from detailed individual lesson plans should only occur when your planning is sound, your co-tutor/mentors agrees and the features of a good lesson plan are still accessible to you and others (see 3.4.4 Audience). **You must always complete the mathematics tasks that you are giving to students.**

3.2.3 Time on planning and review

Initially, try not to spend any longer than 1 hour planning each 1 hour lesson, with 20 minutes for the review. However, even when you come more proficient (e.g. medium term planning first - see 3.3.2 above), you should be spending at least 20 minutes planning for each hour because, as a minimum, you will be completing the tasks from multiple perspectives. You should have 15 minutes review of each hour because, as a minimum, you will be examining the students' work and annotating your plans/resources. Remember, the review time could include your reflective journal entries, subject knowledge audit and planning for your co-tutor/mentor meeting. (If you integrate these activities into your daily/weekly routines then they are both less time consuming and more effective).

3.2.4 Audience

There are two types of audience for your lesson plans: internal (your thinking and lesson aid) and external (co-tutor/mentor, class teacher, ITE coordinator, university tutor, external examiner…).

Any lesson that is formally observed should have a lesson plan and review written for an external audience (i.e. where your rationale could be understood by others).

Initially, all your plans/evaluations should be written for an external audience (as you are likely to be sharing these with the class teacher), but as the course progresses you should reach the point where you can develop your own ‘shorthand’. So, w*ith your co-tutor/mentor’s agreement,* you could write the plans for non-observed lessons in more flexible ways (but you should be able to explain your shorthand if/when asked).

3.3 Key Considerations: Intent, Implementation, Impact

3.3.2 Key Considerations when Planning

Whether consciously deliberative reasoning or more intuitive thought, every time you plan, teach and review lessons you draw on your notions about the nature of mathematics, the learning of mathematics and wider societal/educational issues. All of which are complex and can be viewed from many different perspectives. For example, your decisions about the sequencing of activities may be influenced by your knowledge of the topic, your understanding of variation theory and your beliefs on motivation. Moreover, these three perspectives could be replaced by many more, such as your views on the value of discussion, your beliefs about 'ability' or your stance on how knowledge is created and shared. Each theory or model has the potential to provide powerful insights, but no one single perspective will fully capture the complexity of learning – acknowledging and drawing on different perspectives is a key element in developing your understanding of mathematics and its learning.

When **planning**, consider:

What mathematics* do you want the students to learn? *(Intent)*
Include mathematical concepts/reasoning in addition to skills

How are you going to help this happen (what learning opportunities are you going to create)? *(Implementation)*

How will you know what has been learnt? (Assessment) *(Impact)*

*You need to articulate the mathematical concepts involved, which is harder than it looks (e.g. understanding the role of an = sign) and well as activities/skills (doing). Articulating a skill (e.g. can solve quadratic equations by factorising) may be necessary but is ***not*** sufficient.

Your language needs to be **precise** and your explanations **concise,** this means **scripting** key points and **storyboarding** your board work.

Remember: Many lesson objectives refer to particular skills, but mathematics is more than content; consider 'ways of working' such as mathematical reasoning and problems solving. The National Curriculum offers some good starting points.

E.g. National Curriculum: Fluency, Reasoning, Problem Solving + Content

Plan for your actions *and* student activities. Think about the difference between planning what the students are going to **do** compared to what you want them to **learn**.

3.3.2 Key Considerations when Reviewing (Evaluation) Lessons

When **reviewing**, consider:

What mathematics did you want the students to learn? *(Intent)*
This must include mathematical concepts/reasoning and not just skills (doing…)

What learning opportunities did you create? *(Implementation)*
Who did most work? Who made mathematical decisions? (you or the students)

What evidence do you have about what has been learnt? *(Impact)*
How are you going to use this information to help your future planning?

3.4 A Starting Point for Planning

First, reread section 2.3.4 Classroom Implementation. Planning is complex, so this is just a starting point for first attempts.

You are aiming at a combination of knowledge, skills and understanding, but as a **starting point:**

What do you want the students to **do** (skills to be performed)?

What **recall**/retrieval of prior learning would be useful/needed?

Resources: Find a standard resource (e.g. from school SoW) that requires those key skills. Many schools will use PowerPoints – remember that it is useful for students to see mathematics being written, and make sure you know where you can write if needed.

Golden rule: Complete all the mathematics students might meet (including different ways of completing, different representations and required prior learning)

Starter: Plan a short recall/retrieval practice of key prior learning

Main: Identify the **minimum** you need to explain in order for students to **start** and be able to complete, say, the first third (do not try and cover everything). Take a standard modelling approach such as 'I do, we do, you do'

Script the key explanations. Spoken can be auto-transcribed; visual such as boards/visualiser can be storyboarded (e.g. annotated PowerPoint notes) work. Aim for:

Precise and **concise**

Language matters, details matter, so think carefully about the examples chosen. Remember you need to **share your mathematical decisions making** (the why, the how, the choices…) as well as the what to do – make sure this is in your script.

In the 'we do' you will want to ask questions – key questions also need to be scripted. You will need to anticipate and plan for student responses, including errors and 'don't know'. So, list some ways to follow-up.

Script (write out) what you want the students to write in their books/on worksheets – make sure your planning includes clear instructions for this.

Identify transitions in the task – i.e. when an additional skills are required or complexity is introduced. A4L: plan to look for student response to these questions/activities and have a whole class activity related to these points 'ready to go' to support students to progress. This could be another concise and precise explanation by you – even better if you could incorporate student responses.

Plan how the students are to get feedback on their work when there is still time to do something with the feedback– e.g. answers for even numbers part way through.

Plan some 'tweaks' to the task that can act as extension activities if needed.

Plenary: This will probably include 'the answers' but think about how you can draw attention to key mathematical focus of the lesson. E.g. compare two questions, one 'easy' and one 'hard' and highlight similar structure alongside what has changed.

(The **scripting** is part of your **rehearsal** – try in empty classrooms or with others if possible).

3.5 Planning for Intent, Implementation, Impact – In Detail

Good classroom practice should include, at all levels, a range of teaching approaches, including exposition, discussion, practical work, investigations, consolidation and practice, problem solving, use of technology etc. that leads to a range of learning opportunities.

You should plan for explanations and discussions in as much detail as written tasks, including developing students' vocabulary and mathematical talk.

3.5.1 Planning for Intent: The Mathematics

What mathematics do you want students **to learn**?

This is where you will need to articulate mathematical concepts and ways of working (mathematical reasoning, justifying, conjecturing, problem solving) alongside performance of skills (including fluency) – harder that it looks…

Syllabi and Schemes of Work

- a What does the departmental scheme of work specify?
- b What do the National Curriculum programmes of study and exam syllabi specify

What does the topic involve mathematically?

Some find a mind map a good approach to thinking about a new topic, which can be annotated as the topic is taught

- a The **key concepts**
  - i Identify key concepts
    - i What are the critical features of those concepts?
    - ii What registers and what representations can be used?
    - iii Examples – what forms the example space
  - ii Common misconceptions
  - iii Barriers
- b Sequencing
  - i Example variation – e.g. what range of examples will the students meet (how much of the example space?); how will attention be drawn to critical features?
  - ii Continuity and progression: Links within the lesson and to prior and future learning, links to other concepts and topics
- c Mathematical Reasoning
  - i Opportunities for conjecturing relationships, generalisation, justification, proof…
- d Multiple representations (conceptual variation)
  - i What are the different affordances of each representation?
    - i E.g. How are representations linked to different facets of a concept
      - (a) e.g. a circle split into equal parts with a part/whole approach to fractions
  - ii How are different representations relatable, how do you transform?
- e What alternative solution strategies are there (procedural variation)?
    - i What are the different affordances of each strategy?

Being told just one 'neat' way (that you use) to solve a problem tends to be ineffective for learning in the longer term (learning is messy, and needs to be so)… so plan to give students access to alternatives.

3.5.2 Planning for Implementation and Assessing Impact: Classroom Activities and Talk

What learning opportunities are you going to create and how? (Implementation)

How will you know what has been learnt – assessment? (Impact)

Some key anchors:

How are the students going to access key ideas and concepts during the lesson (including access for review – e.g. captured and shared on the board).

How is attention going to be drawn to mathematical ideas, concepts, structures and mathematically salient features?

How are you going to interpret student activity in terms of learning and how are you going to respond - i.e. assessment for learning (A4L)

Think about planning for teacher activity, student activity and the mathematics (the didactic triangle).

Teacher Activity

- a Teacher Talk (and writing):
  - i Exposition,
  - ii Modelling - direct explanation, partial (gaps, diagram only,…), silent modeling,… thinking aloud…
- b Class talk (and writing)
  - i Q&A – structured question and answer sessions
  - ii Facilitating student explanations,
- c Listening and observing
- d Responding to student activities
- e Purpose:
  - i Instructions,
  - ii Imparting knowledge,
  - iii Drawing attention to mathematically salient features,
  - iv Feedback for students – indicating 'correctness' of responses/work
  - v A4L – assessment and your response (see next section)
  - vi Scaffolding: Supporting effective engagement in tasks – steering direction of travel, support (reducing parameters, parallel modelling), motivation, B4L, frustration control,… (Anghileri, 2006)

Student Activity

a. Listening
b. Writing – copying, consolidation and practice exercises, investigations and problem solving…
c. Talk – answering questions (asking questions), explanations, self-explanation, discussion (develop student-student as well as teacher-student-teacher talk)
d. Student organisation:
    a. individual - pair - group work
    Pair/group: (i) work could it be completed as individuals but discussion encouraged or (ii) work that requires collaboration

e. Student work – personal or shared: exercise books – large paper - board, visualiser, mini-whiteboard, ipad…

Tasks, Examples and Exercises

No resource is perfect, so do not spend a long time looking (or creating). You are far better off selecting a **standard resource**, and then spending time considering how you can use the resource to best effect. E.g. Interrogating tasks by 'tweaking' to add depth – e.g. drawing attention to critical features, what would happen if…, open-ended element to allow for individual development,… *'Busy' work is not the same as learning…*

You **must** complete any tasks you are going to set the students, anticipating student responses (see below).

1) **What mathematics** is 'in' the task/examples?
   a) i.e. What mathematical learning might be available to the students who complete the task?
   b) Articulate the **links** between the **task** completion (skill?) and **learning** about mathematical **concepts**.
      i) E.g. [specific ⇔ general] If you are using examples to illustrate a general concept, articulate how the students might make the transition to the more general ideas.
      ii) Multiple choice questions with common misconceptions (and immediate elimination) can be an effective way to explore concepts through discussion.
   c) **Details matter** [Selection and Sequencing – **variation theory**]
      i) The questions chosen and their sequencing has a big impact on learning opportunities

         E.g. Consider $17-9=$ $\quad 34-6=$ $\quad 25-8=$
         compared to $17-9=$ $\quad 37-9=$ $\quad 97-9=$
         (adapted from Watson and Mason, 2006)
      ii) Examples - How much of the total '**example space'** students experience matters
   d) Consider alternatives [variation theory]:
      i) What different types of representations can be used
         Which are going to be used, when and why?
         How is attention going to be drawn to links between representations?
      ii) What alternative solution strategies are there?
         Which are going to be used, when and why?
      iii) How are you going to support students to make decisions themselves (e.g. which strategy and/or representations) and evaluate different approaches?
         What criteria will you use/share when selecting approaches?
   e) Anticipate student responses, and your counter-responses (see below)
      i) Anticipate a range of responses, not just the ideal*; include common errors (misconceptions)/barriers and how you might respond.
         *go beyond your 'neat' approach, to the different routes learners might take.
      ii) Plan **scaffolding** – what you could do to support students, if needed, whilst still leaving them access to the key learning points. E.g. (i) articulate what you could take away (e.g. reduce parameters) and what you would leave the students to do. (ii) plan additional parallel models. (iii) plan alternative strategies

iii) Plan ‘deeper’ – what you could do to deepen understanding rather than moving on (see section 2.3.4 for ‘tweaking’).

f) Discussions: You need to plan for any student discussions to the **same level of detail** as a written exercise, and with similar levels of scaffolding/support. Students need more that the instruction to ‘discuss’ to have productive discussions.

2) **Detailed Scripts** (written and verbal)**:** You must plan the mathematics to the level of individual numbers/letters…
   a. Examples
      a. **Numbers matter** – e.g. 2^2 is a poor example to demonstrate squaring…
      b. *So, script your examples*
   b. Written mathematics: Board work
      a. What will be seen, when
         i When will students have access to the key mathematical ideas (or do these disappear after a few seconds when the PowerPoint slide goes or when you run out of board space?)
         ii Will the students see mathematics being written (as opposed to presented)?
         iii *So, map/script/ storyboard your board work*
   c. Verbal mathematics
      a. Your explanations and key questions need to be planned in the same detail as your written examples
      b. *So, script your (planned) verbal explanations and key questions*
   d. Student responses
      a. You should anticipate student responses to verbal questions as well as written tasks
      b. Plan how/ what you will share key ideas with the rest of the class
         i Do not expect students to pick up ideas when they hear them once
         ii *So, script/ storyboard how you will share key ideas with additional/alternative forms of communication and/or recording in more permanent ways so the students can review/refer back.*

The opportunity to review: It is often said that learners need to hear/experience something in two or three ways for even relatively simple issues (e.g. if you are giving page/question numbers: verbal and written on the board as part of classroom routines). Remember, students are unlikely to pick up complex concepts immediately and on the first occasion. This is particularly relevant for verbal information; consider additional ways to **capture key information** so students have opportunities to revisit/review (both your verbal contributions and those from students). This is particularly relevant if you are using PowerPoints, with key information displayed for a limited period of time.

Technology can be very powerful (e.g. dynamic geometry) but ‘try before you buy’.

Planning for Assessment - Impact: Interpretation of, and response to, Classroom Activity

A4L: Assessing students' understanding includes acquiring information to shape the lesson trajectory and to inform future lessons. So, plan what you are going to assess and how *and* what you are going to do with that information (A4L).

> As listed above, the modes of assessment could be listening to students, asking questions, looking at written responses. But you then need to decide what it is you are listening/ looking for, or what you are going to ask.
>
> Then you need to plan for how you will respond. You cannot anticipate everything, but having clear ideas about your key learning intentions will help here.

Where are the learners – explore **student reasoning** – take time on this step

Where do you want them to be – your **mathematical horizon**

What can you do, and more importantly, what can you get the students to do to move in this direction

Plan opportunities to explore student reasoning to balance attempts to move students towards your 'mathematical horizon'. In particular, if a student does not provide your desired answer to a question (an 'error'), consider options other than immediately steering then towards your 'correct' solution. Unpacking their thinking will enhance your assessment as well as provide opportunities for students to hear/discuss other students ideas – if facilitated, the students may be able to untangle issues for themselves and have ownership of the solution.

Anticipated Student Responses:

A key part of planning is the anticipation of student responses, as this enables you to plan:

> Assessment and response strategies - including scaffolding and adding depth.

You should anticipate a range of possibilities, not just your desired answers. With anticipated responses, your plans can then include ideas about what you could do if particular errors appear that still leaves the substantive work with the students.

> For example, you may have identified key (pivot) questions that introduce a new element/feature. Your anticipated responses might include an anticipated error and students 'getting stuck'. You plan ideas about how you can respond with options other than you 'telling'. You can look for student responses for these key questions and depending on numbers, you could help individuals/small groups or bring the class back together.

This anticipation is difficult when you are new to teaching, but your lesson reviews will allow you to develop this aspect of your pedagogical knowledge. Students will, hopefully, always be able to surprise you, but anticipating will help you plan for learning and assessment.

Scaffolding

Scaffolding is support mechanisms for effective student engagement in tasks, but the key is to leave the mathematical work with the students. Scaffolds should not be permanent, but fade over time. Scaffolds can focus on cognitive engagement, keeping student heading in the right direction, frustration control, and enlisting students' interest. Alternatively, scaffolding could simplify the task by reducing parameters or by offering a parallel model (e.g. Wood, 1976). The key is that learning objectives should still available to the student, rather than be completed by the teacher. Drawing attention to critical features is another form of scaffolding if these were missed by students (Anghileri, 2006).

It is all too easy to do the work for the students when they appear to be stuck. Leaving the substantive work with the students and support that fades over time are key elements of scaffolding.

Interpreting student responses in terms of learning.

How are you going to interpret student activity in relation to learning? I.e. you need to plan how you are going to assess students' learning (a complex undertaking).

Where are they – explore student reasoning – take time on this step Where do you want them to be – your mathematical horizon What can you do, and more importantly, what can you get the students to do to move in this direction?

In lesson approaches:

a. Listen, watch
   i With a 'good' activity, you will usually gain more information from students unprompted activity or student (self) explanation than from direct teacher questions, so identify critical activities/ questions and plan how to collect and review information.
b. Questions and discussions
   i Include a range of response mechanism e.g. technology, mini whiteboards…
   ii Move beyond single solution questions – e.g. if I changed… what would happen; which is the odd one out and why; (see Watson and Mason Thinkers)
   iii Remember to plan how to explore student reasoning before 'moving' students to your mathematical horizon (your reasoning)

Post-lesson:

c. Review work, talk to students, homework, assessments

Note: Students can provide a 'correct' (wanted) answer without understanding the underlying principle; students can provide an 'incorrect' (unwanted/unclear reasoning) answer whist having a sound understanding of the underlying principle.

Remember: Mathematics is more than content, so alongside assessing students' capacity to produce a correct single answer, also think about how you can assess reasoning, justification, ways of working mathematically (e.g. checking strategies), … The more you can encourage the students to talk and discuss their mathematics (using mathematical language accurately) the easier it is to assess their understanding.

3.5.3 PowerPoints

Spending a long time writing **PowerPoints**? **Stop…**
Presenting already completed mathematics is not the same learning experience as writing on the board or a visualiser, where the students see mathematics being constructed. Writing PowerPoints can take a long time and can produce inflexible lessons that might be unresponsive to students' activities and needs. Indeed, key mathematical ideas may only be visible for a short period of time. So, consider if writing on a whiteboard or visualiser (or

combining PowerPoints and handwriting) during lessons would be quicker, offer more flexibility and allow students to see mathematical processes in action. (You do, however, need to spend time *planning in detail* what you are going to write/ask). As above, also think about how students can have access to key mathematical ideas throughout the lesson, so they can go back and review key (novel) ideas and link to later work – this can be hard with PowerPoints.

Consider: Writing on a board or visualiser ‘by hand’ can be both efficient and effective. Use PowerPoints judiciously - e.g. if the technology allows, projects graph paper, which you can then draw on by hand.

An annotated PowerPoint is almost invariably not the same as a lesson plan (this may be different to other subjects). You need to ensure that you have considered student learning alongside your activities, and that all features of the lesson plan pro forma are covered. If student activity indicates alternatives routes are preferable you should have enough flexibility to move away from any PowerPoint.

3.6 Implementation: Teaching – In Detail

Remember: It is all about getting the students to do the mathematics…

Minimise, as far as possible, you doing mathematics and maximised student activity

'Perfect' teacher explanations do not automatically mean good learning experiences – whist 'telling' has a key role, 'telling/explaining all' does not work. If students do not understand, more teacher talk often leads to more confusion. Aim for the **minimum explanation**, rather than the maximum 'covering every aspect' version, **so the students can start** doing mathematics themselves.

Remember: **Precision** and **Concision**

Revisiting key ideas once students have tackled some work and met some of the issues will allow for fuller, more accessible explanations.

Moreover, mathematics is full of complex concepts and learning is messy, so students will need to revisit to grapple with ideas in different ways in order to gain understanding.

So, plan carefully what you are going to say/show (**script** & storyboard); but often, what works well is just enough of you so the students can 'get started'. When the students have completed some work, you can then revisit and discuss some of the more complex aspects, as the students have a reference point to work from and they can contribute.

Reflection-in-Action is hard, so try and build in time for you to watch and listen – very legitimate A4L activities.

Remember

Remember: Do mathematics, think about mathematics, discuss mathematics

Spend time on the mathematics (thinking, doing, consideration of alternative approaches and representations, questions to ask, and anticipating student responses).

Remember: Details matter; you need to plan specific examples and questions, down to the level of individual numbers, e.g. considering the impact of changing $3^5 \div 3^2$ to $3^4 \div 3^2$ or $3^5 \div 3^5$ or $3^2 \div 3^5$or $3^5 \div 4^2$…

Remember: It is better to use a standard resource/task that you have thought about in detail, rather than spend a long time looking for the 'perfect' resource.

Ask: Where is the mathematics? Who is doing the mathematics?

Teacher explanations are a key part of teaching mathematics, but students' experiences should go beyond practicing methods that you have shown them.

Ask: "Who have I planned for… myself or the learners?"

You need to think about what you are going to do, but only as a vehicle for generating learning opportunities for students. So, remember to plan student activity.

Remember: The key is to get the students to do the mathematics, not you!

Remember: With student responses, we cannot assume a direct link between the 'correctness' of an answer and understanding… it is possible that

'correct' answers ≠ understanding & 'incorrect' ≠ misunderstanding

Remember: Try to understand student reasoning before moving them to your version. Their reasoning may not be immediately apparent or logical from your perspective, but it will be part of their mental models (and rational to them). Invariably, those models have been built predominantly in mathematics classrooms, so any misconceptions are more ours than theirs.

3.7 Impact: Review Each Lesson – In Detail

When **reviewing**, consider:

What mathematics did you want the students to learn? *(Intent)*
This must include mathematical concepts/reasoning and not just skills (doing…)

What learning opportunities did you create? *(Implementation)*
Who did most work? Who made mathematical decisions? (you or the students)

What evidence do you have about what has been learnt? *(Impact)*
How are you going to use this information to help your future planning?

All lessons must be **reviewed** (evaluated), and you should also review each topic when completed. However, how you review will evolve over time, as described below, but you must always **talk about mathematics**.

3.7.1 Review (Evaluation) Processes: Describe⇨Interpret⇨Review

This links to Observation in Section 3.8:

Describe (in detail) – 'account of' Interpret (explicit reference to pedagogical reasoning and theoretical frameworks) – 'account for' Review (your understanding of the lesson, your pedagogical content knowledge ,…) [Evaluate – make judgments]

Lesson/topic review (evaluation) can take a number of forms:

- Annotating a lesson plan and/or resources can be very effective (and efficient)
- A (short) written commentary (exemplar prompts on Blackboard and in section 5)
- Reflective entries
- Discussions with co-tutor (mentor)/class teacher (inc. preparation for co-tutor/mentor meetings)

If the lesson has been formally observed by your co-tutor (mentor)/the class teacher then a more detailed written review would be expected.

There can be a tendency for student teachers to:
Focus on general levels of student behaviour/engagement
Talk about '*the class*…' in general terms
What went 'wrong'

These activities can feel like lesson reviews (evaluations), and some discussion of behaviour may be relevant, **but** they rarely offer insights into learning that will help your development.

So please **remember to discuss the mathematics** in detail - in order to consider the learning of mathematics you will need to be **specific** – the mathematics and particular students.

E.g. you need to consider particular questions and responses. The precision of language and terminology are important (e.g. hearing "two four" rather than "two to power four" for 2^4 could be important). You cannot talk about the learning of 'the class' without articulating what individual students have said, done, written….

Consider the range of evidence available to you (your observations, students' work, observation feedback, assessment information, talking to students…). Over time, undertake a range of activities:

- Articulate in detail what happened during a key episode
- Focus on particular students/groups of students
- Focus on particular elements of the lesson (maybe linked to your targets).
- For a sequence of lessons (e.g. topic review), consider tracing the progress of a few students through lesson evaluations, scrutiny of their work (including homework), topic tests, talking to them…

3.7.2 Lesson Debriefs

For your observed lessons you will have the added benefit of feedback from your co-tutor(mentor)/class teacher. If there is time between the lesson and debrief you should prepare by undertaking your own review first. You should have mutually agreed a focus for the observation, so make sure you address that aspect of the lesson.

There can be a tendency for student-teachers to focus on:

Behaviour Engagement Mathematics

We would like a more balanced view:

Behaviour Engagement Mathematics

Which we will encourage by asking your co-tutor/mentors to:

Ensure your lesson plans articulate mathematical concept development alongside teacher and student 'activity'.

E.g. Explicitly stating how tasks link to mathematical concepts

Steering you to talk about mathematics.

E.g. Ensure ERFs and debriefs contain some discussions about specific examples of mathematics from the lesson.

To use 3.7.1 structure: Describe (in detail); Interpret (explicit reference to pedagogical reasoning and theoretical frameworks); Review; [Evaluate]

3.7.3 Debrief Prompts for Co-tutor/mentors

The following are some prompts discussed at co-tutor/mentor meeting that could help in developing a shared understanding of what may have happened in particular lessons. Please add to as appropriate.

Remember 3.7.1: Describe (in detail); Interpret (pedagogical reasoning); Review; [Evaluate]

1. Do you recall … part of the lesson.

   A Describe what the students were doing [describe]

   B (If response A has no mathematics) Describe what mathematics the students were doing

   C What mathematical learning might have been happening? [interpret]

      a e.g. Practising a skill (developing fluency); understanding a concept

   A What did you notice [describe] and what was happening [interpret]

   B How did you respond [describe] and can you articulate your reasoning? [interpret]

   C What have you learnt about your pedagogical reasoning [review]

2. What was your rationale for… (task/activity) <intention>.

    E.g. Concepts and possible misconceptions; possible alternative approaches and representations

    What did you think would happen; what did happen [describe] and why [interpretation] <implementation>, (how do you know <impact>)

3. Examples/exercises

    Why those numbers/examples, *why that sequence* <intention>

    What did you notice [describe] <implementation>;

    Generalisation from examples: how might students make the step from examples to more generalised concepts and what gaps/misconceptions might occur from those particular examples – what is the possible impact on students' example spaces [review]

4. Talk to me about (student A or a group of 2-3 students). Describe their engagement with… [describe] and what this might indicate about their understanding/learning today [interpret].

5. Ask

    A What mathematics did you want the students to learn <intent>

    Mathematical concept development *<the understanding>* as well as skills/fluency *<the 'doing'>*

    B What were the key learning opportunities, and were they expected or unanticipated? If you 'deviated' from your plan, why? <implementation>

    C What evidence do you have (or can get) that will help you assess what may have been learnt <impact>

6. Where was the mathematics?

    E.g. Where was the mathematical decision making, with you or the students?

    E.g. Was mathematical decision making an implicit or explicit part of the lesson

    E.g. Was it all 'show' by the teacher followed by 'practice' by the students

    E.g. How many different opportunities did the students have to gain access to complex ideas – such as, review of key ideas from other perspectives, examples that challenge the boundaries, multiple representations, alternative solution strategies…

3.8 Observation and deconstruction: Of others (or your own reviews) ⇔ Teaching

Observing other teachers' lessons, and reviewing your own, is harder than it looks and what you look for will change what you notice. It is important to try to distinguish between classroom activity and your interpretation of that activity (which draws on your models of learning, be that explicitly or implicitly). The CCF also identifies some specific elements where you *learn how to* by "Observing how expert colleagues … and deconstructing this approach" Consider stages:

Focus

Articulate your focus

Describe

Describe what is said, written, done in as much detail as possible (be specific about the mathematics – e.g. copy the board). Remember, you only notice a limited amount of what happens (your focus, beliefs and values all influence what you attend to).

Interpret

Articulate what the observations *might* indicate about:
(i) Student learning – be tentative and be careful not to make assumptions. *This is where you should be using **mathematics education research**, periodicals and wider curriculum resources to interrogate your observations, as this interpretation will draw on your models of learning. Articulate your pedagogical reasoning.*
(ii) Other 'hidden' aspects of classrooms, if appropriate, such as classroom norms (e.g. routines), teacher decision making, student engagement or student motivation. As with learning, be tentative and articulate your reasoning drawing on sources.

Review and reflect

Reflect on your understanding of the lesson (for observations of others see activities listed below) and implications for your pedagogical practice.

[Evaluation: Evaluation is making judgments about effectiveness – this is not required and can be counterproductive if undertaken without careful consideration of the steps above first. Remember, if you are observing qualified colleagues, review is important (deconstructing with expert colleagues is an integral part of the CCF) but you would be stepping beyond the bounds of a student-teacher if *you* evaluate *them*. So think carefully when you talk the class teacher to ensure that you remain professional and do not overstep the bounds of a student teacher. Equally, evaluating your own practice may be 'inaccurate' and focus on unproductive elements; review and reflection tend to be more productive.]

3.8.1 Possible Observation Activities:

You must discuss observation protocols with your co-tutor/mentor before observing lessons.

The regular class teacher is likely to be happy for you to act as a teaching assistant. This can be a useful, but remember to plan your observations so you undertake a *range of activities*. Be active when observing; **details matter**, so take detailed notes of what is written, said and done… and **be specific about the mathematics**. The following are some possible approaches:

1. Recreate a lesson plan
    a. What have you identified as the mathematical focus?
        i. If possible, link to the wider medium-term plan
    b. What is the relationship between the tasks and the lesson aims (see 2)
    c. Map out the conceptual journey:
        i. What is the mathematical journey in the lesson and beyond?
        ii. E.g. required prior knowledge; conceptual development during the lesson; links/contribution to future learning
2. Tasks: All involve completing the tasks set for students.
    a. Articulate what conceptual development/learning opportunities are embedded in the tasks set for students
    b. Complete 'as a student' and compare your approach with students' work.
    c. Complete with as many different approaches and representations as you can think of.
    d. Identify possible barriers/ misconceptions
    e. Identify assessment and feedback opportunities.
3. Observe a small group of students (focus on the students rather than the teacher) – take details notes of what they say, do, write – be very specific, especially about the mathematics.
    a. What might these actions say about the students' learning?
4. Consider who is involved in the mathematics and how. For example:
    a. Map out who is involved in mathematical discussions and the form
        i. E.g. what type of vocabulary is used and by whom
        ii. E.g. are 'full sentences' used by students?
    b. Map out who is involved in mathematical decision making
        i. e.g. who selects the method for solving a problem
    c. Make notes about what written mathematics was shared with students and how.
    d. Are there issues of equity of access / social justice that you have noticed (take care with language, assumptions and judgments).
        i. E.g. Are there particular groups that have greater/less involvement?
5. Consider the relationship between 'skills' and the 'concepts'.
    a. E.g. identify skill acquisition (e.g. solving one step linear equations) and how this interacts with potential concept development (e.g. what 'solving' means, understanding the different roles of letters, different types of equations, and when 'something' is solvable, the range of possible approaches…)
6. Focus on particular lesson features – describe and interpret.
    a. E.g. Routines: Transitions; integration of administrative activities;
    b. E.g. Class talk: structure of discussions; questioning
        i. Make notes about who speaks, what questions were asked, by whom; how/who answered; how were responses followed up (was there a difference between 'correct' responses and 'errors'?).
        ii. Was an IRE pattern dominant?
        iii. Feedback
    c. Scaffolding
    d. Sequencing
    e. Assessment

3.9 Some Key Features of 'Effective' Lessons

Different styles can be equally effective, but observation and research have identified key features that commonly recur in successful lessons.

Preparation and Planning

- Students' prior understanding is taken into account (e.g. examine students' books, ask…; articulate anticipated student responses)
- Mathematics has been thought through from a variety of angles (alternative approaches, alternative representations, potential misconceptions, links to other topics, prior learning requirements)

Lesson Beginnings

- The teacher is punctual and students are admitted in an orderly but welcoming way
- Lessons start promptly, with students engaged in a meaningful activity

Relationships

- Consistency: the teacher is 'seen to be fair' by students
- The teacher shows an interest in students, but is neither permissive nor intent on seeking popularity for its own sake
- The teacher conveys enthusiasm for the topics and activities

Voice, Instructions and Explanations

- The teacher voice is clear, audible and 'non-monotone'
  - The teacher does not speak over students.
- Instructions are clear, without total reliance on verbal instructions (e.g. if there is a page number, it is written on the board as well as verbally stating)
- Language is mathematically accurate, with ambiguity avoided (**precision**)
  - Complex concepts do need to be discussed, with students encouraged to use mathematical language accurately and appropriately. So do not avoid mathematical terms, but take care how they are used, especially if new.
- Consider how students are going to access explanations, remembering they are unlikely to pick up key ideas on the first verbal exchange or PowerPoint slide. So **concise** and **precise** explanations **plus** provide longer terms 'access', such as well-planned board work.

Classroom presence

- The whole classroom is the teacher's realm (i.e. not welded to the desk)
- The teacher is aware of what individuals are doing in the lesson
  - e.g. overview by 'scanning', standing at the back, time for observing/listening
- Eye contact and other non-verbal communications are used for minor misdemeanours
- When the above fails and problems arise, intervention is prompt and at the lowest level possible
  - If the lesson runs into break/lunch 'keeping back' individual or small groups of students for a few minutes to discuss expectations can be an effective way to improve behaviour, but avoid whole class punishments. Certainty is more impactful than severity of penalties – so two minutes at break can work wonders – though work within school policies!

Appropriateness of Topics and Activities

- Work is appropriate to the ages, prior learning and cultural background of the students
- Lessons offer a variety of activities and teaching methods

Maintenance of Purposeful Activity

- Transitions are organised to avoid 'dead time' between one activity and another
- When students finish before others, there are clear protocols on what to do next

Lesson Endings

- Highlighting significant mathematical ideas met in the lesson is often appropriate
- Students should have some means of assessing their learning (though this is likely to occur throughout the lesson)
- If required, homework requirements are explained clearly and in good time
- **Finish on time**
    - Materials and equipment are put away efficiently, with the room left tidy
    - Students are dismissed in a controlled and orderly manner

You can do all the above and still have a student(s) misbehave. However, good organisation and good mathematical activities influences student perception of the teacher and the lesson, awakens interest, holds attention and reduces the chances of disruption.

4. Algebra Equivalence Task

Overview

These activities are designed to prompt your thinking about underlying mathematical structures and to raise your awareness of the pedagogy (learning and teaching) of mathematics. Whilst all the activities are presented to you on paper, there are suggestions as to how these might be used with students. There are tasks to complete, after which there are opportunities to reflect on the underpinning mathematical concepts and how these might be made available to learners. Depending on your mathematical background, you may need to vary the amount of time you spend on the mathematical content (there are optional elements) and the pedagogical foci.

Once you have completed the activities there is a commentary that outlines the design principles. No tasks are perfect – please interrogate and make your own suggestions.

Focus: Equivalence

This session focuses on how diagrams can be used to explore the equivalence of algebraic expressions.

Mathematical content: The equivalence of linear and quadratic expressions.

Pedagogical content: Multiple representations (affordances and limitations, critical features and links between features and concepts).

There are three activities – you do not have to complete all of the activities – spend time focussing on problems appropriate to you.

Activity 1: Algebraic Expressions Card Sort

Aims: To explore links between algebraic expressions and area representations.
To recognise equivalent expressions, and to understand why they are equivalent.

Activity 2: Dissecting Diagrams

Aims: To translate between algebraic expressions and area representations.
To draw on the structure of the area representations to explore what happens in more general cases.

Activity 3: Completing the square

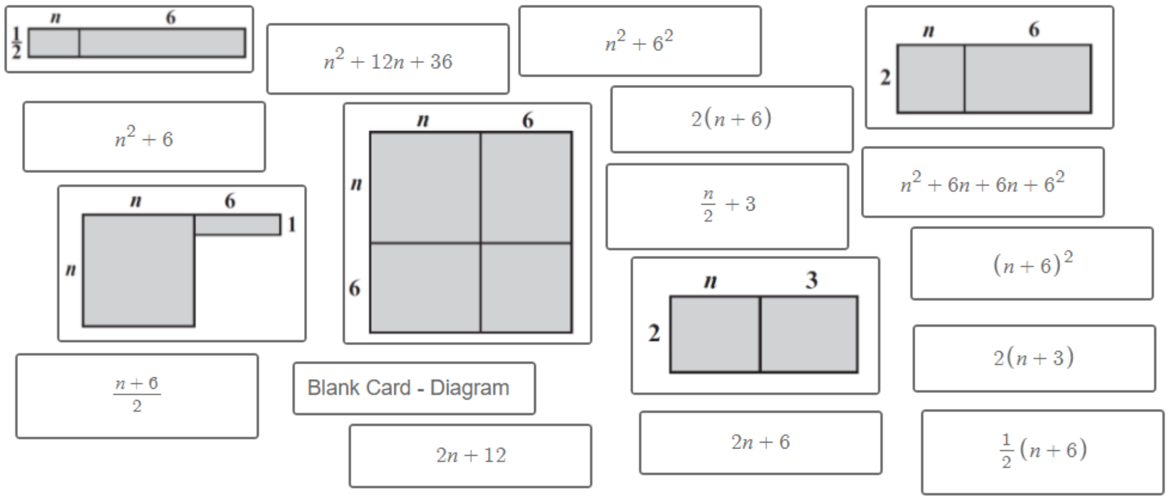
Aim: To use area representations of expressions to develop an understanding of completing the square.

There are some open exploratory questions, so please use discuss and share ideas with your peers. As well as this being an opportunity to engage with other people's ideas, you can ask questions and/or request feedback on your thinking.

Activity 1: Algebraic Expressions Card Sort

[This could be presented on Desmos or an equivalent online platform where card sorts can be dragged/snapped to form groups.]

Match the expressions with the appropriate diagram. Draw your own diagram(s) if needed.



[For students a solution rubric could be available on completion/request – at end of section]

Reflection on Activity 1

Initially, select one group that contains more than one expression and respond to the following prompts. Then consider if anything changes when you look at the other groups.

a) Expressions:

All the expressions in one group are **equivalent**. In your own words, explain what this means and why this is the case.

Consider: What values can n take? What is the role of '=' in expressing equivalence?

b) Diagrams:

Explain how the expression(s) are linked to the diagram. Try to be specific and link specific features of the diagram to elements of the expression(s).

Consider: Are there any issues/limitations in using diagrams to represent expressions?

c) Further examples:

Consider: Are there other diagrams or expressions that could represent your chosen group? If so, how many and of what form?

Which expressions are likely to be 'preferred' by, say, a teacher?

Other Perspectives on Activity 1

On the following page are *some* examples of responses for the $2(n+3)$ group. How do these ideas relate to your reflections – do they prompt any additional thoughts? Can you express any of the ideas more clearly or precisely using mathematical terminology?

[In classrooms, you could replace the following with the students' ideas. Rather than present all together, a sequence of vignettes could be used – which would you choose and why?]

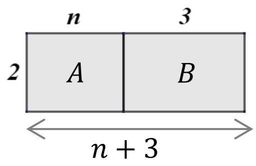
Expressions:

Equivalence means the expressions represent the same value.

Multiplication is distributive over addition, so

$$2(n+3) = 2 \times n + 2 \times 3$$
$$= 2n + 6 \qquad \text{where } n \text{ can be any number}$$

Diagrams:



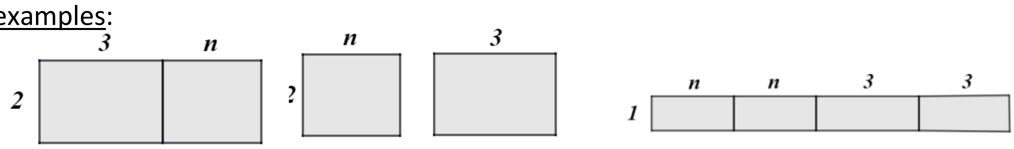
Lengths: $n, 2, 3$ & $n+3$ are the lengths of sides of A, B and/or the combined rectangle.

Area: $2n$ is the area of rectangle A and 6 is the area of B. Similarly, $2(n+3)$ and $2n+6$ are the area of the combined rectangle.

Equivalence: Area is conserved (i.e. area of combined rectangle = area of A + area of B). Therefore, the two **expressions must be equivalent**, i.e. $2 \times (n+3)$ is the same as $2 \times n + 2 \times 3$.

Potential Issues: Use of integers (and n) could prompt integer only thinking. Modelling as length would imply n is positive. Scale – in these diagrams n 'looks' smaller than 3. Illustrating subtraction appears to be difficult.

Further examples:



You could represent the area in an infinite number of ways…

$2(n+3) = 2n+6$ are the two 'preferred' formats – the shortest following writing conventions.

Conventions: removal of $\times$ and write the term with the highest power first.

Others $(n+3) \times 2 \quad 2 \times (3+n) \quad (3+n) \times 2 \quad 2(3+n)$

$2 \times (n+3) \quad n \times 2 + 3 \times 2 \quad 2 \times n + 2 \times 3 \quad n \times 2 + 6$…

If you include examples such as $2n + 4 + 2$ or $3n - n + 6$, there are infinite possibilities.

Equivalent because 1) same area 2) all can be rearranged into the same form using properties of numbers and operations. Distributive and commutative properties:

E.g. commutative ($+$ *and* $\times$) $\quad 2n + 6 = 6 + 2n$ and $2 \times (3+n) = (3+n) \times 2$

E.g. $\times$ distributive over $+$ $\quad 2(n+3) = 2 \times n + 2 \times 3$

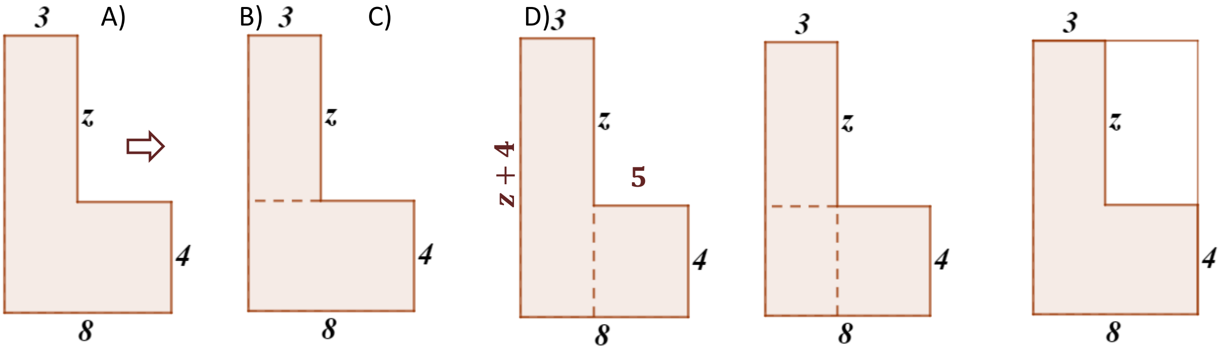
REVIEW: If you feel you are not fully conversant with the commutative, associative and distributive properties for arithmetic operations ($+ \; - \times \div$), this would be an appropriate time to review this element of your subject knowledge.

SELF-EXPLANATION: Write a short explanation about how linear and quadratic expressions can be represented by diagrams. If possible, draw attention to equivalence and the different features of linear diagrams compared to quadratics.

Activity 2: Dissecting Diagrams

A standard way to find the area of compound shapes is to split them into simpler shapes and add (or subtract) components; this can be done in different ways. If one or more side length is represented by a letter, then equivalence of algebraic expressions can be explored.

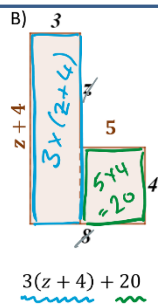
For example, an 'L' shape can be split into two or three rectangles, or an encasing rectangle can be formed with the excess area subtracted.



Area calculations: $3z + 32$ $3(z + 4) + 20$ $3z + 12 + 20$ $8(z + 4) - 5z$

Form an expression for the area based on each dissection. Match elements of the calculations to appropriate rectangles in each diagram and check the equivalence of the expressions.

[For students, a full explanation rubric would be available on request – see end of section]



Questions to explore (you can be selective):

What would happen if…

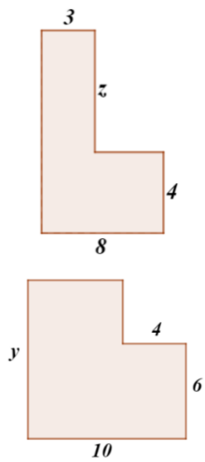
The numbers changed

Are there any constraints on the numbers that can be used?

The letter was in a different position

Does the type of algebraic manipulation required to show equivalence change?

Are there any constraints on how the letter can vary?

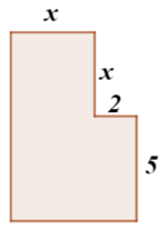


Generating diagrams from expressions:

Can you illustrate $5n - 20$ with an 'L' diagram – if so, is it unique?

Can you illustrate any linear expression with a diagram?

Cases with the same letter in perpendicular dimensions:



What expressions are generated with different dissections of this diagram?

Check that the expressions are equivalent.

How is this similar or different to the previous examples?

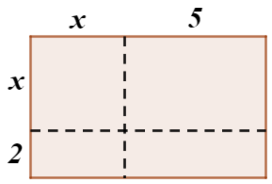
What would happen if the numbers changed or the positions of the letter changed?

Can you illustrate $x^2 + 9x + 20$ as an 'L' shape?

Can any quadratic be illustrated with an 'L' shape?

If yes, how? If no, which expressions can be illustrated and what are the constraints?

Rectangles: A common representation of a quadratic is a complete rectangle.



What equivalent expressions can be generated from this diagram?

Can all quadratics be illustrated with a rectangle?

E.g. can expressions with subtraction be represented?
How about coefficients of x^2 other than 1

Hint: You could start by looking at a particular family of quadratics, such as 'the difference of two squares'. Start by exploring an example, such as $x^2 - 9$, then move onto a general case, $x^2 - a^2$.

Remember: Identify the type and range of values a letter can take. For example, is the letter restricted to positive integers, or to any integer, or can it take any real value, or…

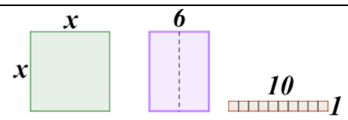
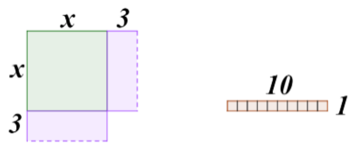
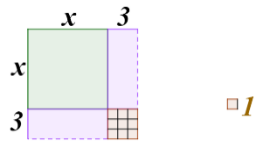
Activity 3: Completing the square

Rearranging quadratic equations from $y = ax^2 + bx + c$ form into $y = a(x + d)^2 + e$ can be useful (e.g. it is easier to find roots and turning points).

Students can find this rearranging difficult, so some advocate the use of diagrams to help understanding. However, diagrams can create extra complexity for the learner if they are not understood. As you work through the task, consider if the previous two activities could provide suitable familiarisation.

[A responsive model could be made available on Geogebra, or similar, for $\left(\frac{b}{2}\right)^2 < c$]

Write a narrative, explaining the process and the links between the algebra and the diagrams.

Algebra	Diagrams (illustrating 'completing the square')	Narrative
$y = x^2 + 6x + 10$		
$= x^2 + 3x + 3x + 10$		
$= x^2 + 3x + 3x + 9$ $+ 1$ $= (x + 3)^2 + 1$		

Consider: What happens when you try to take the same dual algebraic/diagrammatical approach with:

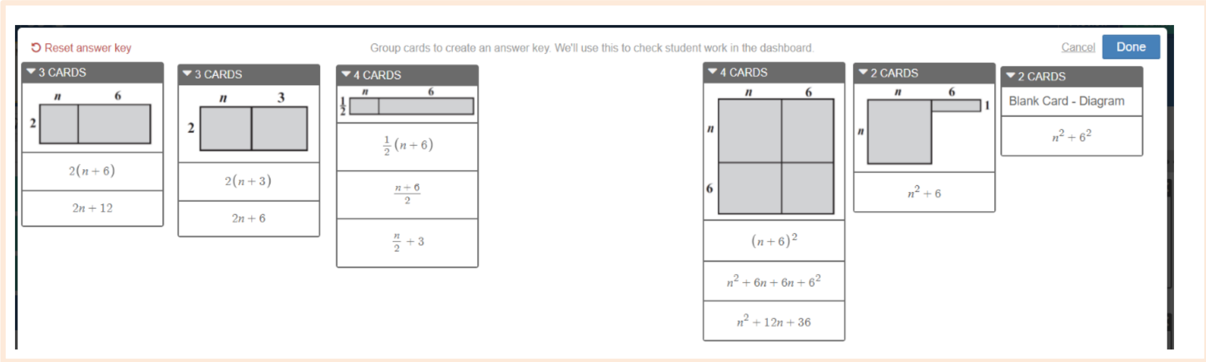
1. $y = x^2 + 6x + 8$ 2. $y = x^2 - 6x + 10$ 3. $y = 2x^2 + 6x + 10$

Can you adapt the diagrams (and/or the algebra) to take account of these differences?

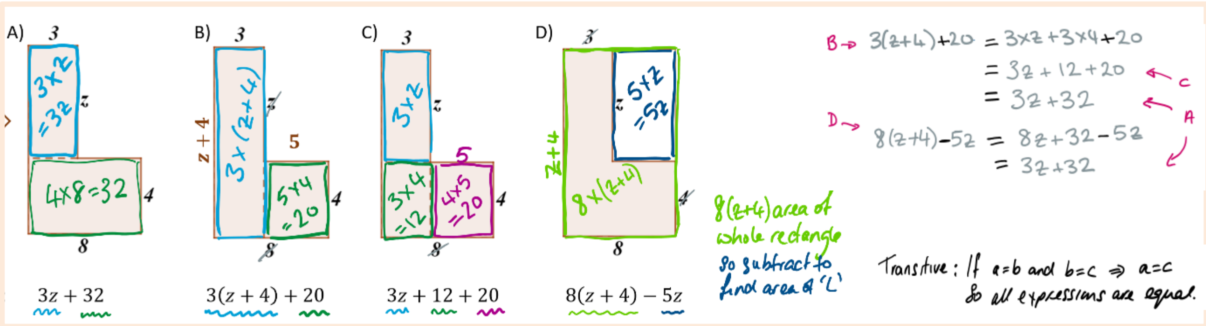
Can you rearrange any quadratic into a 'completing the square' format?
I.e. how far can you generalise and what, if any, are the constraints.

Solution Rubrics

Activity 1



Activity 2



Commentary: Design Principles

This commentary that outlines the design principles that underpin this task.

Context

The materials are intended for a Mathematics Subject Knowledge Enhancement course for adults due to join a Secondary Mathematics PGCE course (Initial Teacher Education). These courses are designed for students without a mathematics related degree in order for them to develop the required subject knowledge. As a minimum, they would have passed GCSE mathematics at grade C/4, but most would have a higher mathematics related qualification. As such, the students should have met the mathematical content of the course, but there is likely to be a large range of proficiency and understanding. In addition to revising and developing the students own subject knowledge, the courses aim to prompt thinking about how others learn mathematics. As such, the tasks are intended to be flexible, allowing students to shift to a pedagogical orientation, especially on topics with which they are more familiar.

Acknowledgments

The activities draw on materials designed by others. For example, the Standards Unit (Malcolm Swan) forms the basis of Activity 1 and I observed the approach used in Activity 2 in a secondary classroom.

Theoretical Concept: Equivalence.

The fundamental role that equivalence holds in mathematics means that a ‘flexible understanding of symbolic equivalence relations is essential for a conceptual understanding of many branches of mathematics’ (Jones et al., 2013, p.34). However, research indicates that many learners develop an operational rather than a relational understanding of the equal sign, which has a negative impact on their algebraic reasoning (e.g. Fyfe et al., 2020).

Here, the intention is that area representations of algebraic expressions can provide a justification for the equivalence of algebraic expressions. This use of diagrams allows the learners to access the notion of equivalence before they are fully conversant with rearranging expressions. In particular, this offers ways to explore sameness and substitution relationships, which are subsequently captured by the equal sign. This should offer opportunities for students to develop a relational understanding of equivalence.

Any diagrammatical representation has its limitations, and may imply constraints that do not apply more generally. The intention here is for these visible constraints, such as lengths being positive, to prompt students to explore the bounds within which they are working in a more explicit manner.

Multiple Representations

Research indicates that multiple representations of a concept have a paradoxical role; simultaneously being a key mechanism for developing understanding, whilst also constituting an area of profound difficulties for many learners (Duval, 2006; Dreher and Kuntze, 2015). Rau and Matthews (2017) argue that asking students to articulate links between representations, and between representations and concepts, aids understanding. Here the intention is to draw students’ attention to critical features, such as the role of area in equivalence, whilst also exploring potential issues, such as scale.

Design Principles

To provide learning opportunities that allow learners to build coherent concept images, thereby providing a framework within which they can develop more formal deductive reasoning (Tall, 2002).

> One key element is the structured use of examples, so learners can identify properties and non-properties, extending their example space to see the general through the particular (Watson and Mason, 2002). Variation theory is one mechanism that can be used to analyse the mathematics made available to learners through the examples met (Kullberg et al., 2017).

To structure tasks that support the development of a mathematical disposition, including metacognitive knowledge and self-regulatory skills (De Corte et al., 2000).

> This should include opportunities for students to reflect on tasks and their learning. Feedback is also an integral part of these processes. In these activities, this includes self-explanation tasks, where the goal is to make reasoning more explicit, and the scaffolding and fading of prompts related to generalisation.

A specific response to "a lesson without learners having the opportunity to generalise is not a mathematics lesson." (Mason, Graham and Johnston-Wilder, 2005, p. ix) is made at the end of this document.

Commentary

There are three related activities. The expectation would be that students select from these activities, rather than complete all activities and questions.

Activity 1: Algebraic Expressions Card Sort

Aims: To explore links between algebraic expressions and area representations.
To recognise equivalent expressions, and to understand why they are equivalent.

Card Sort (Standards Unit A1): The mixture of linear and quadratic expressions allows for attention to be drawn to their different properties. Multiple solution strategies allows the task to be accessed with different skills sets. The repeated use of the digits 2, 3 and 6 reduces matching through non-mathematical relationships, and should prompt a closer examination of the expressions. With only one fraction in the card sort, these might be matched through visual similarity. As fractions tend to be more problematic for some students, this could provide a way to introduce fractional coefficients.

The subsequent reflective task draws attention to key features, in case these were not prompted by the initial card sort. The solution rubric and the other perspectives provide additional support for those students with weaker subject knowledge.

Alternative next steps:
Students could be asked how they might draw diagrams for expressions that included subtraction.
The use of 'errors' can be an effective way of exploring concepts and misconceptions; students could be asked to identify, explain and correct errors presented in a similar card sort.

Activity 1: Reflection

Depending on how familiar students are with this type of reflection, notes may be limited or extensive.

The shift to all groups, after an initial focus on one group, is intended as a first step in moving from a particular case to more general properties. Equivalence is stated; this is implied by the card sort but explicit reference is made to ensure both the language and concept are transferred. The diagrams are relatively simple, with connections to the expressions relatively direct. The idea is for students to work in a familiar context to develop the habit of identifying critical features of diagrams and to articulate how these features relate to concepts. The expressions in the card sort are all in their simplest form. The further examples provide a checkpoint for those not fully conversant with rearranging expressions. This also provides an opportunity for students to consider the boundaries of their examples, which could extend their example space.

Activity 1: Other Perspectives

These are offered as a mechanism to draw attention to key features that students may not have noticed or recognised as important. Equivalence is emphasised, and specific issues with the diagrams are highlighted, as explicit discussion of critical features of diagrams is often absent (Rau et al., 2017).

REVIEW: Anecdotal evidence indicates that many students have an intuitive understanding of commutative, associative and distributive properties.

SELF-EXPLANATION: There is some research evidence to suggest that self-explanation can support understanding, and in particular the relationships between representations (Berthold and Renkl, 2009; Rau and Matthews, 2017). However, effects are far from consistent and appear to be dependent on a range of factors, including the quality of scaffolding provided (Rittle-Johnson et al., 2017). Consequently, self-explanation tasks are unlikely to be effective in isolation.

Activity 2: Dissecting Diagrams

Aims: To translate between algebraic expressions and area representations.
To draw on the structure of the area representations to explore what happens in more general cases.

The compound 'L' shape dissection offers a systematic way to generate equivalent expressions, both linear and quadratic. The CHECK and explanation rubric offer support for students more unfamiliar with the content or approach. The 'questions to explore' offers examples of expressions and diagrams as starting points, with prompts to consider translation in both directions. The 'what would happen if…' questions start with more local changes, before leading to more general cases. This is designed to support students in extending their example space, whilst also considering constraints.

The two different diagrams for quadratics provide opportunities for students to consider the critical features of diagrams (e.g. the same letter in perpendicular dimensions) and constraints, required or assumed (e.g. integer values).

Activity 3: Completing the Square

Aim: To use area representations of expressions to develop an understanding of completing the square.

For students more familiar with quadratics, the exploration of completing the square provides a flexible problem that can be explored at a variety of levels. The narrative section prompts the students to make explicit the links between the algebraic process and the features of the diagram and its manipulation. The further examples draw attention to the affordances and limitations of the diagrams, and hence their critical features. This could also allow the students to recognise the general algebraic process that could be applied independent of diagrammatical support. The final generalisation could include the formulation of the 'quadratic formula'.

Generality: Response to Mason, Graham and Johnston-Wilder (2005).

Mason et al. (2005) argues that generalisation underpins all areas of mathematics, and they discuss how generality can be met in a variety of contexts, from number to diagrams. They highlight that there are different levels of generality; for example, $y = 2x + 3$ has one level of generality, as x and y vary, but this is subsumed within $y = mx + c$ that covers all points on a plane. Here, opportunities to generalise come in a variety of forms. There is the notion that a single expression, such as $2n + 6$, captures an infinite family of equivalent expressions. The structural difference between diagrams representing linear and quadratic expressions could be discerned, with opportunities to see how these differences manifest in expressions. There are prompts, such as 'what would happen if…', with the intention to move thinking beyond the initial example. There are specific questions prompting generalisation, such as 'can any ... expression be represented' and 'can all … expressions be represented'. (Though it should be noted that Mason et al. (2005) offers a cautionary note about the ambiguity of language. Here, 'any' may cue sequential thinking about examples, where 'all' may cue a more holistic view about generalised properties.)

References

Berthold, K. & Renkl, A. (2009). Instructional Aids to Support a Conceptual Understanding of Multiple Representations. *Journal of Educational Psychology,* 101**,** 70-87.

De Corte, E., Verschaffel, L. & Op't Eynde, P. (2000). Self-Regulation: A Characteristic and a Goal of Mathematics Education. *Handbook of Self-Regulation.* Elsevier, 687-726.

Dreher, A. & Kuntze, S. (2015). Teachers' Professional Knowledge and Noticing: The Case of Multiple Representations in the Mathematics Classroom. *Educational Studies in Mathematics,* 88**,** 89-114.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational studies in mathematics,* 61**,** 103-131.

Fyfe, E. R., Matthews, P. G. & Amsel, E. (2020). College Developmental Math Students' Knowledge of the Equal Sign. *Educational Studies in Mathematics,* 104**,** 65-85.

Jones, I., Inglis, M., Gilmore, C. & Evans, R. (2013). Teaching the Substitutive Conception of the Equals Sign. *Research in Mathematics Education,* 15**,** 34-49.

Kullberg, A., Runesson Kempe, U. & Marton, F. (2017). What Is Made Possible to Learn When Using the Variation Theory of Learning in Teaching Mathematics? *ZDM,* 49**,** 559-569.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing Thinking in Algebra,* London: Sage.

Rau, M. A., Aleven, V. & Rummel, N. (2017). Supporting Students in Making Sense of Connections and in Becoming Perceptually Fluent in Making Connections among

Multiple Graphical Representations. *Journal of Educational Psychology,* 109**,** 355-373.

Rau, M. A. & Matthews, P. G. (2017). How to Make 'More' Better? Principles for Effective Use of Multiple Representations to Enhance Students' Learning About Fractions. *ZDM,* 49**,** 531-544.

Rittle-Johnson, B., Loehr, A. M. & Durkin, K. (2017). Promoting Self-Explanation to Improve Mathematics Learning: A Meta-Analysis and Instructional Design Principles. *ZDM,* 49**,** 599-611.

Tall, D. (2002). The Psychology of Advanced Mathematical Thinking. *Advanced Mathematical Thinking.* Springer, 3-21.

Watson, A. & Mason, J. (2002). Extending Example Spaces as a Learning/Teaching Strategy in Mathematics. *In:* Cockburn, A. & Nardi, E., (eds.) *26th Conference of the International Group for the Psychology of Mathematics Education (PME26)*, Norwich, UK, 21-26 July. PME, 4-377.

5. Exemplar Plans

5.6.1 Example: Medium term plan

Please see Blackboard for further examples of short and medium terms plans.

The value of using such topic overviews is that it encourages you to consider links between concepts, representations and approaches and how to develop understanding over time. This should help you be flexible and be responsive to student contributions. For example, if students grasp a particular concept quickly you could challenge their understanding by allowing them to explore counter-examples or pull forward ideas from later lessons. If students are struggling you can map alternative routes to the key concepts.

The following is an example of the start of the process of linking a medium-term plan to individual lessons. From this medium-term structure individual plans are then written
LO = Learning Objective **KC** = Key Concepts **CMisC** = Common Misconceptions

Shape, Space and measures 1

Lesson 1 **LO**: Develop students' knowledge of notation and labelling conventions for lines, angles and shapes. Review measuring of angles.

KC: Angles: a measure of turn (arbitrary- 360 around a point). CMisCs: size of angle related to length of lines; misuse of protractor esp two scales; measuring errors

Activities: labelling points, lines and angles quiz that includes multiple choice questions with known common misconceptions

Resources: Impact Mathematics 2G p. 36-40

Lesson 2 **LO**: Angle sum of a triangle and derive angle sum of a quadrilateral

KC: Proof vs examples; angles in any triangle sum to 'half a turn'; any quadrilateral can be split into two triangles => angles sum to 360. CMisC: mathematical proof

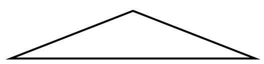
Activities: students draw a triangle and use a protractor to measure the angles (discussion of class results inc. measurement errors, 'proof' from examples); link to ripping corners, tessellating triangles and dynamic geometry demonstration. Split quadrilaterals into two triangles to derive angle sum

Resources: Worksheet -triangle sum and quadrilateral sum

Lesson 3 **LO**: Understand a proof that the exterior angle of a triangle is equal to the sum of the interior opposite angles

KC: Knowledge of exterior angles; proof (algebra and 'seeing' triangle as representing all triangles rather than the one pictured). CMisC: Exterior angle drawn incorrectly; non-standard orientation of triangle => misidentification of exterior angle

Activities:



In pairs How can we work out angle x? Does this work for all triangles (ask to link back to tessellating triangles if prompt needed)? Construct dynamic geometry representation.

Resources: worksheet – exterior and interior angles in a triangle; Geogebra

---

Lesson 4 **LO**: To be able to solve angle problems with parallel lines

KC: Angle properties; switching between specific ⇔ general. CmisC: lack of reasonableness check

Activities: Worksheet with 2 parallel lines and a transversal (in groups of three with three angles), students to measure all angles and identify equal angles and compare with others in their group. Questions: What is the relationship between the angles? How can you move the lines so the angles will stay the same; so the angles change in size but relationships remains; what happens if… (the lines move) **Resources:** key stage 3 mathematics p. 190-191
Lesson 5 **LO**: To be able to solve angle problems **KC:** Focusing on different elements of a diagram at different stages (ignoring different elements and/or adding to the diagram e.g. extending lines). CMisC: relating general rules ⇔ specific examples **Activities**: Angle problems with multiple stages. Students to find as many different ways to solve as possible, describing the 'rules' that they use at each stage. Reverse questions (describe the rules used and students to fill in angles) **Resources**: Impact Mathematics 2R p. 29-33
Lesson 6 **LO**: To develop understanding of angle facts, in particular 'checking techniques'. **KC:** Mathematical reasoning, estimation, generation and manipulation of diagrams **Activities**: angle dominoes, multiple choice quiz with common misconceptions, students to write own revision booklet, students to construct dynamic geometry demonstration of an angle rule **Resources**: key stage 3 mathematics p. 192-194; Geogebra

5.6.2 Example: Medium Term and Short Lesson Plan – with prompts

Class Information

Year: Class: Lessons: Homework:

Students:Seating plan: Prior Attainment: SEND/EAL:

Medium Term Plan				
Year :	**Class:**	**Topic :**	**Start Date:**	**Duration:**

Topic Objectives	**Key Concepts**
Articulate what we want the students to learn. Consider "Students will: "be able to do..." **and** "understand that..." and even "know..."	Articulate the key mathematical concept(s) that are at the heart of these lessons (this could be done as a mind map) Include common misconceptions Consider links forward and backwards

Language:

Representations:

Techniques (solution strategies):

Contexts:

Lesson (session) Objectives	**Content: Main Activities**	**Lesson (session) Concepts** Examples/tasks to Concepts	**Depth and Support**	**Assessment Opportunities**	**Resources**
Shorthand version taken from topic objectives Consider ways of working as well as content: e.g. **NC** Reason mathematically: interpret when the structure of a numerical problem requires additive, multiplicative or proportional reasoning	The main activities (including any key examples to be used). *Identify: alternative solution strategies; alternative representations ; anticipated student responses (inc common misconceptions)* Consider ways of working: E.g. Opportunities for group/ pair/ individual work. Teacher and student use of IT. Alternative ways to record and share work.	Shorthand version taken from key concepts and misconceptions *Identify: how the activities give access to key ideas.* *E.g. Examples: 'what is' & 'what is not...';* *'intelligent variation';* *range of permissible change (e.g. integer only answers?)* *Identify: how activities might highlight and address misconceptions*	*Identify: how students who grasp the main concept(s) quickly will be challenged and how students who initially struggle will be supported*	Opportunities to gather information (both you and students) How could assessment information be used Feedback opportunities for students – esp. other that direct teacher feedback	Specific resources (e.g. p246 from ...)
All mathematical tasks should be completed – highlighting: *); how activities give access to key ideas; alternative solution strategies, alternative representations, a range of anticipated student responses (inc. misconceptions)...*					

<table>
<tr><th colspan="4">Lesson Plan</th></tr>
<tr><th>Year:</th><th>Class</th><th>Topic:</th><th>Lesson:</th></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="4">Personal Target(s):</td></tr>
<tr><td colspan="4">In lesson prompts</td></tr>
<tr><td colspan="4">Outline plan
Starter:
Main:
Key Questions (plenaries):</td></tr>
<tr><td colspan="4">Individual Learners:
Identify specific approaches/actions needed to support particular students in this lesson.</td></tr>
<tr><td colspan="4">Post lesson prompts</td></tr>
<tr><td colspan="4">Evaluation: Be specific (and talk about mathematics!)
Annotated lesson plans can be used but for some lessons, such as those observed, a longer written evaluation in addition to your annotations would be appropriate.
Assessment: What do I think the students learnt (and on what evidence am I making my judgement). (Consider individual students – what do you know?)
Were there any unforeseen events?
Did anything work particularly well (how do you know)? What might you change next time and why?
Consider varying grain size – e.g. sometimes look at particular elements of the lesson/ particular students, so you can evaluate in more detail, rather than the whole lesson in more general terms.
Prompts: Relationships, Engagement, Explanation, Transitions, Other adults, Questioning, Assessment, Differentiation, Handling of misconceptions, Organisation of students, Task design.</td></tr>
</table>

5.6.3 Example: Lesson Plan

Lesson plan

<table>
<tr><td colspan="5">Class: Date: Lesson:</td></tr>
<tr><td colspan="5">Personal T&L Targets: From dialogue with (co) tutors, links to teaching standards</td></tr>
<tr><td colspan="5">Topic:
Links: backwards and forwards</td></tr>
<tr><td colspan="3">Learning Challenge: May be found in medium term plan
Lesson objectives: (What the students will learn)
Consider ways of working as well as content related
You might not share all of these with students; you could bold those to share.</td><td colspan="2">Outcomes: (What the students will achieve)
(take care to consider learning rather than ‘doing’ as a measure of success)</td></tr>
<tr><td colspan="5">KC (Key Concepts):
Articulate the key mathematical concept(s) that are at the heart of this lessons
Include: Alternative representations and CMisC Common Misconceptions
For UA1 & UA2, this should include references to research
Skills</td></tr>
<tr><td>NC (SoW):</td><td colspan="4">Resources:
Articulate how tasks enact concepts (this information may be on annotated resources); how activities give access to key ideas; alternative solution strategies, alternative representations…</td></tr>
<tr><td colspan="5">Starter: [There may be occasions where the lesson is not split into starter/ main]
Practice of fluency skills (e.g. number manipulation), a way of working mathematically or a task that will support the main activity.</td></tr>
<tr><td colspan="2">Main: Teacher Activities
[Exposition, modelling, Q&A, discussion, task introduction, students’ contributions (verbal, board/visualiser)...]
Examples, exercises/tasks: what will be used; links to key concepts (inc. examples ⇔ generalisation); sequence (variation theory – links between questions).
How will you draw attention to the main concepts?
Alternative representations/ approaches - how used?
A4L: e.g. Listening, key questions (written, verbal), mini plenaries…</td><td colspan="2">Student Activities
What are the students going to be doing?
[Listening, answering verbal questions, practicing, investigating, problem solving…]
Where is the mathematical learning (e.g. listening ⇒ exercise to practice a skill, or task with the potential to develop understanding)?
Students working individually, in pairs or in groups?
Medium used; class board/ visualiser, IT, exercise books, large paper?
How students share ideas with peers.</td><td>Anticipate student responses
This may be recorded on with the resources, such as on worksheets.
What different approaches might be used… including possible misconceptions.

How will students receive feedback on their progress (this does not have to be directly from the teacher).</td></tr>
</table>

Differentiation: *e.g. ways of working, 'rich' task with multiple entry points/routes/solutions (low floor/high ceiling), methods of presentation, feedback systems.* *'Scaffolding': e.g. how parameter might be reduced, though remember to articulate what elements of learning remain with the student and what you are* temporarily *removing/undertaking for them.*
Plenary: *More than a summary: could themes be drawn together, could the concepts be considered from an alternative perspective, what are the boundary conditions for the topic?*
Individual learners: *EAL, SEND, others, (please take care wrt PP: unless you can articulate how you are going to adapt the lesson to meet a* learning *need please consider if it appropriate to include in a lesson plan)*
Homework:

Evaluation: *This would usually take the form of an annotated lesson plan, but for some lessons, such as those observed, a longer written evaluation in addition to your annotations would be appropriate.* *Prompts: Relationships, Engagement, Explanation, Transitions, Other adults, Questioning, Assessment, Differentiation, Handling of misconceptions, Organisation of students, Task design.*

Extract from lesson plan on proportional reasoning.

Topic: Proportional Reasoning | Links: Multiplicative relationships

Learning Challenge: Students to develop understanding of how to identify situations with multiplicative relationships and hence use proportional reasoning to solve problems. To use and discuss their own methods for solving problems using proportion. **Lesson Objective:** Students to explore the difference between enlargement and non-proportional stretches and to develop an understanding of the difference effects of additive vs multiplicative changes. To relate to other contexts.	Outcomes: To associate enlargement with multiplicative relationship with picture 'looking correct'. To relate multiplicative relationships to other contextual contexts, such as paint remaining the same colour. To appreciate that there are more than one approach to solving a multiplication problem

KC (Key concepts)
Understanding the **structure** of the question: Identifying when there is a proportional relationship**;** the multiplicative relationship and **the nature of what remains constant.**
Two types of multiplicative relationships: 'scale factor' and 'functional factor'
Skills: manipulation of the multiplication and division relationships and the associated calculations (including 'easy' transition points (partial additive approaches) and unitary method as appropriate.
CMisC:
Misidentification of proportional relationship
Not recognising what is considered as the whole/ 100% / base unit.
Multiplication v additive relationships (including when partial additive approach is suitable)
Over-relying on 'spotable' relationships: unsure about when/what to × and ÷ (and what), especially when have fractions or decimals- Not able to use flexibly the unitary method vs easy transition points.

Curriculum: Identified by assessments/class teacher as potentially problematics, with a tendency to have a mixture of additive and multiplicative calculations.
Research: Swan (2005), Hodgen et al. (2009) Brown et al. (2010): Key findings- misidentification of relationships, impact of 'simple' examples, inappropriate contexts…

Stater: (i) 15 x 12 = (ii) 3 x ? = 12 (iii) 4 x ? = 6 (iv) 8 x ? = 18 (v) ? x 1.5 = 12
(vi) how many different multiplication calculations can you find with the answer …
[with integers only- consider using prime/non-prime; fractions…]
Q: What happens if…; Would ? be; Can you work backwards…;
(**short-**may choose to start with A) Adapted multiplication grid: Initial grid (consider 4x4) with direct multiplication (consider smaller grid size so multiplication other than x2 can be included). Inverses and none integer multiplication / solutions in remaining questions. Open ended element to help manage late arrivals etc. e.g. what different rules could fit … what would happen if 1 was changed to a… what would happen if 4 was changed to b

Main: Teacher Activities	**Student Activities**	**Anticipate student responses**
A. Provide students with pictures of enlargements/ np stretches (constant: visual image of a circle) Questions Which relationship(s) 'maintain' the picture, which do not? Would this happen for any x	Students to explore the effect of additive and multiplicative changes based on visual image of a circle	np stretches and enlargements identified. relate np stretches to additive relationship relate enlargement to x2 x1.5 relate enlargement to 'add itself, add on half'

…

References

Anghileri, J. (2006). Scaffolding Practices That Enhance Mathematics Learning. *Journal of Mathematics Teacher Education,* 9**,** 33-52.

Bakker, A., Smit, J. & Wegerif, R. (2015). Scaffolding and Dialogic Teaching in Mathematics Education: Introduction and Review. *ZDM,* 47**,** 1047-1065.

Baldry, F. (2019). *Teachers' Orchestration of Mathematics for Different Groups of Students.* PhD, University of Leicester

Ball, D. L. (1993). With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics. *The elementary school journal,* 9**,** 373-397.

Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special? *Journal of teacher education,* 59**,** 389-407.

Berthold, K. & Renkl, A. (2009). Instructional Aids to Support a Conceptual Understanding of Multiple Representations. *Journal of Educational Psychology,* 101**,** 70-87.

Cobb, P., Stephan, M., McClain, K. & Gravemeijer, K. (2011). Participating in Classroom Mathematical Practices. *In:* Yackel, E., Gravemeijer, K. & Sfard, A. (eds.) *A Journey in Mathematics Education Research.* Dordrecht: Springer, 117-163.

Confrey, J., Maloney, A. P. & Corley, A. K. (2014). Learning Trajectories: A Framework for Connecting Standards with Curriculum. *ZDM,* 46**,** 719-733.

Craig, C. J. (2018). Metaphors of Knowing, Doing and Being: Capturing Experience in Teaching and Teacher Education. *Teaching and Teacher Education,* 69**,** 300-311.

De Corte, E., Verschaffel, L. & Op't Eynde, P. (2000). Self-Regulation: A Characteristic and a Goal of Mathematics Education. *Handbook of Self-Regulation.* Elsevier, 687-726.

Dreher, A. & Kuntze, S. (2015). Teachers' Professional Knowledge and Noticing: The Case of Multiple Representations in the Mathematics Classroom. *Educational Studies in Mathematics,* 88**,** 89-114.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational studies in mathematics,* 61**,** 103-131.

EEF (2021). *Cognitive Science Approaches in the Classroom: A Review of the Evidence* UK: Education Endowment Foundation.

Fyfe, E. R., Matthews, P. G. & Amsel, E. (2020). College Developmental Math Students' Knowledge of the Equal Sign. *Educational Studies in Mathematics,* 104**,** 65-85.

Hmelo-Silver, C. E., Duncan, R. G. & Chinn, C. A. (2007). Scaffolding and Achievement in Problem-Based and Inquiry Learning: A Response to Kirschner, Sweller, And. *Educational Psychologist,* 42**,** 99-107.

Jerrim, J., Austerberry, H., Crisan, C., Ingold, A., Morgan, C., Pratt, D., Smith, C. & Wiggins, M. (2015). Mathematics Mastery: Secondary Evaluation Report. *Education Endowment Foundation*.

Jones, I., Inglis, M., Gilmore, C. & Evans, R. (2013). Teaching the Substitutive Conception of the Equals Sign. *Research in Mathematics Education,* 15**,** 34-49.

Kilpatrick, J., Swafford, J., Findell, B. & council, N. r. (2001). The Strands of Mathematics Proficiency *In:* Kilpatrick, J. & National Research Council (eds.) *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academy Press.

Kirschner, P. A., Sweller, J. & Clark, R. E. (2006). Why Minimal Guidance During Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-Based, Experiential, and Inquiry-Based Teaching. *Educational psychologist,* 41**,** 75-86.

Knowledge Quartet (2020). *What Is the Knowledge Quartet* [Online]. Available: http://www.knowledgequartet.org/.

Knuth, E., Zaslavsky, O. & Ellis, A. (2019). The Role and Use of Examples in Learning to Prove. *The Journal of Mathematical Behavior,* 53**,** 256-262.

Küchemann, D. & Hoyles, C. (2005). Pupils' Awareness of Structure on Two Number/Algebra Questions. *Proceedings of European Research in Mathematics Education IV***,** 438-447.

Kullberg, A., Runesson Kempe, U. & Marton, F. (2017). What Is Made Possible to Learn When Using the Variation Theory of Learning in Teaching Mathematics? *ZDM,* 49**,** 559-569.

Mason, J. (2011a). Explicit and Implicit Pedagogy: Variation Theory as a Case Study. *In:* Smith, C., (ed). *British Society for Research into Learning Mathematics*, Oxford, UK, 19 November. BSRLM, 107-112.

Mason, J. (2011b). Noticing: Roots and Branches. *In:* Sherin, M. G., Jacobs, V. R. & Philipp, R. A. (eds.) *Mathematics Teacher Noticing: Seeing through Teachers' Eyes.* New York: Routledge, 35–50.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing Thinking in Algebra,* London: Sage.

NCETM (2017). *Five Big Ideas in Teaching for Mastery* [Online]. Available: https://www.ncetm.org.uk/teaching-for-mastery/mastery-explained/five-big-ideas-in-teaching-for-mastery/.

Peters, E. (2008). Numeracy and the Perception and Communication of Risk. *Annals of the New York Academy of Sciences,* 1128**,** 1-7.

Prestage, S. & Perks, P. (2013). *Adapting and Extending Secondary Mathematics Activities: New Tasks for Old*: David Fulton Publishers.

Rau, M. A., Aleven, V. & Rummel, N. (2017). Supporting Students in Making Sense of Connections and in Becoming Perceptually Fluent in Making Connections among Multiple Graphical Representations. *Journal of Educational Psychology,* 109**,** 355-373.

Rau, M. A. & Matthews, P. G. (2017). How to Make 'More' Better? Principles for Effective Use of Multiple Representations to Enhance Students' Learning About Fractions. *ZDM,* 49**,** 531-544.

Rittle-Johnson, B., Loehr, A. M. & Durkin, K. (2017). Promoting Self-Explanation to Improve Mathematics Learning: A Meta-Analysis and Instructional Design Principles. *ZDM,* 49**,** 599-611.

Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary Teachers' Mathematics Subject Knowledge: The Knowledge Quartet and the Case of Naomi. *Journal of Mathematics Teacher Education,* 8**,** 255-281.

Rowland, T., Thwaites, A. & Jared, L. (2015). Triggers of Contingency in Mathematics Teaching. *Research in Mathematics Education,* 17**,** 74-91.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational studies in mathematics,* 22**,** 1-36.

Shulman, L. S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational researcher,* 15**,** 4-14.

Skemp, R. R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics teaching,* 77**,** 20-26.

Straesser, R. (2007). Didactics of Mathematics: More Than Mathematics and School! *ZDM,* 39**,** 165-171.

Stylianou, D. A. & Silver, E. A. (2004). The Role of Visual Representations in Advanced Mathematical Problem Solving: An Examination of Expert-Novice Similarities and Differences. *Mathematical Thinking and Learning,* 6**,** 353-387.

Sweller, J. (2021). Why Inquiry-Based Approaches Harm Students' Learning. *CIS Education Program.* NSW Australia: The Centre for Independent Studies.

Sweller, J., van Merriënboer, J. J. & Paas, F. (2019). Cognitive Architecture and Instructional Design: 20 Years Later. *Educational Psychology Review***,** 1-32.

Sweller, J., Van Merrienboer, J. J. & Paas, F. G. (1998). Cognitive Architecture and Instructional Design. *Educational psychology review,* 10**,** 251-296.

Tall, D. (1995). Cognitive Growth in Elementary and Advanced Mathematical Thinking. *PME conference*. PME, 1-61.

Tall, D. (2002). The Psychology of Advanced Mathematical Thinking. *Advanced Mathematical Thinking.* Springer, 3-21.

Van Es, E. A. & Sherin, M. G. (2002). Learning to Notice: Scaffolding New Teachers' Interpretations of Classroom Interactions. *Journal of Technology and Teacher Education,* 10**,** 57.

Watson, A. (2017). Pedagogy of Variations: Synthesis of Various Notions of Variation Pedagogy. *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories.* Brill Sense, 85-103.

Watson, A. (2019a). Comments on Sweller, Van Merriënboer & Paas' 2019 Cognitive Load Update.

Watson, A. & Mason, J. (2002). Extending Example Spaces as a Learning/Teaching Strategy in Mathematics. *In:* Cockburn, A. & Nardi, E., (eds.) *26th Conference of the International Group for the Psychology of Mathematics Education (PME26)*, Norwich, UK, 21-26 July. PME, 4-377.

Watson, A. & Mason, J. (2006). Seeing an Exercise as a Single Mathematical Object: Using Variation to Structure Sense-Making. *Mathematical thinking and learning,* 8**,** 91-111.

Watson, S. (2019b). Revisiting Teacher Decision Making in the Mathematics Classroom: A Multidisciplinary Approach. *In:* Jankvist, U. T., Van Den Heuvel-Panhuizen, M. & Veldhuis, M., (eds.) *Eleventh Congress of the European Society for Research in Mathematics Education (CERME11)*, Utrecht, Netherlands, 6-10 February. Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Xu, W. & Zammit, K. (2020). Applying Thematic Analysis to Education: A Hybrid Approach to Interpreting Data in Practitioner Research. *International Journal of Qualitative Methods,* 19**,** 1609406920918810.